Lecture 4.

n distinct objects
select k of them.

- order matters ?
- with / without replacement.

Case I. order matters without replacement.

k steps :

step 1: n

step 2: $n-1$

$\vdots$

step k : $n-k+1$

Total way $=$ $\underline{n(n-1)\cdots(n-k+1)}$.

Def (permutation) The number of ways to select k objects out of n distinct objects. where order matters and no replacement.

$$A_n^k := n(n-1)\cdots(n-k+1).$$

read as "n permute k".

$1, 2, 3, \ldots, n.$

② ①, 3. ...... n.

③, 2, ①, ... n.

$$A_n^n = n!$$

$$A_n^k = \frac{n!}{(n-k)!}$$

Case 2: order does not matter. without replacement.

$$A_n^k = \underbrace{\boxed{(\text{\# order does not matter})}}_{?} \times k!$$

Step 1. Step 2.

$$? = \frac{A_n^k}{k!} = \frac{n!}{k!\,(n-k)!}$$

(Definition) The above number, denoted by

$$\binom{n}{k} := \frac{A_n^k}{k!} = \frac{n!}{k!(n-k)!}$$

is called the Combination number, which is the # ways to select k from n distinct objects where order does not matter & no replacement

Case 3. order matters, with replacement

k steps   step 1: n

step 2: n.

step k: n

Total $= n^k$. k^n ✗.

Case 4. order does not matter, with replacement.

$n=4$ △ □ ○ ☆. △, ○, △ k selections

$k=3$ 2 0 1 0 4 △ 0

how many times each object is selected.

The i-th object was chosen x_i times $(x_i \geq 0)$.

x_1, x_2, x_3, x_4

2 0 1 0

$$\boxed{\sum_{i=1}^{n} x_i = \boxed{k}}$$

n objects $\Rightarrow x_1, \dots, x_n$

Total # selections = solutions to above eqn.

The balls-and-bars model.

① o o | ② o o | ③ o | ④ o | ⑤

$x_1 = 2$, $x_2 = 2$, $x_3 = 1$, $x_4 = 1$. $x_5 = 0$

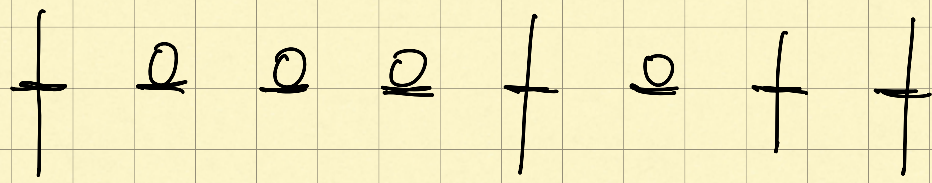
k balls.

$n-1$ bars.

$k+n-1$ positions

select k from $k+n-1$

$$\binom{k+n-1}{k}$$



$k+n-1$

	replacement ✓	replacement ×
order ✓	n^k	A_n^k
order ×	$\binom{n+k-1}{k}$	$\binom{n}{k}$

Poker : answering the lecture 1.

	Probability.
Royal Flush.	0.000154 %
Straight Flush.	0.001385 %
Four of a kind	0.02401 %
Full house	0.144058 %
⋮	⋮

?

Classical probability. 52 cards.

order ×

replacement ×

$$\binom{52}{5} = \frac{52 \times 51 \times 50 \times \cancel{49} \times 48}{1 \times 2 \times 3 \times 4 \times 5} \qquad \frac{n!}{k!(n-k)!}$$

$$= 2{,}598{,}960.$$

size of sample space.

$\mathbb{P}(\text{Four of a kind}) = \dfrac{624}{2,598,960}$.

Step 1: Choose the rank : 2 ~ Ace.

13 ways.

A	A	A	A
♡	◇	♣	♠

52.

Step 2: Choose the remaining 5-th card

$52 - 4 = 48$.

Total $= 13 \times 48 = 624$.

Classical Probability.

- $|S| < \infty$.
- outcomes equally likely.

Geometric Probability.

- S is uncountably infinite (but S has "nice" shape).
- outcomes equally likely.

↓

easy to compute Area(S).

Def (Geometric probability).

If S has a "nice" shape. For $A \subset S$.

$$\mathbb{P}(A) := \frac{\text{Area}(A)}{\text{Area}(S)}.$$

$$\mathbb{P}(w_i) = \frac{1}{\infty} \quad \times.$$

e.g. Two persons X, Y. meet at the library. 12:00.

X arrives between $\underline{11:50} \sim 12:10.$ (0)

Y $11:55 \sim 12:15.$

Both X, Y waits for 5 mins.

What is the probability X, Y meet?

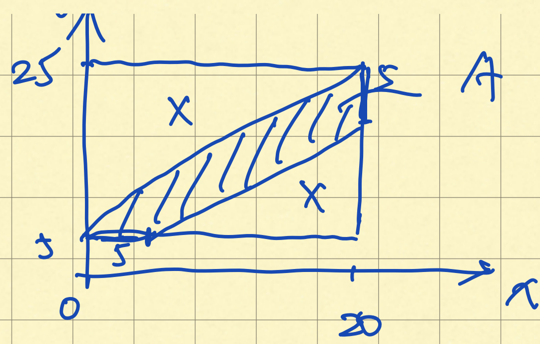
$$S = \left\{ (x, y) \;\middle|\; \begin{matrix} 0 \le x \le 20 \\ 5 \le y \le 25 \end{matrix} \right\}. \qquad \text{Area}(S) = 400.$$

A = event that X, Y meet

$$= \{ (x, y) \in S \mid |x - y| < 5 \}.$$

$$\mathbb{P}(A) = \frac{\text{Area}(A)}{\text{Area}(S)} = \frac{400 - 200 - 50}{400} = \frac{150}{400} = 0.375.$$

y


25
5
0
20
5
5
X
X
A
x