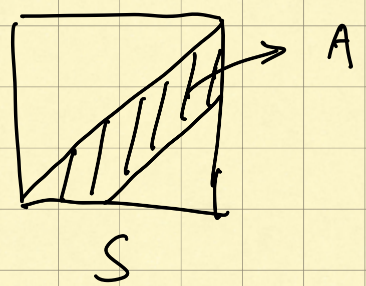
Lecture 5.

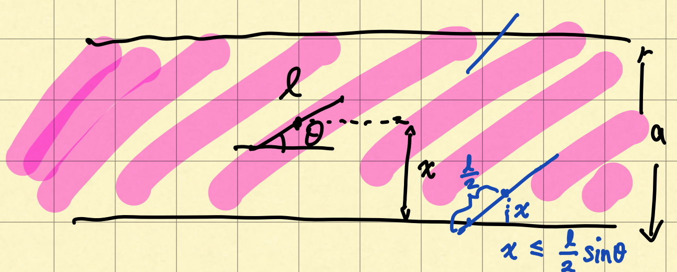
- S has a "nice" shape
- $w_i, w_j \in S$

$$\mathbb{P}(A) := \frac{\text{Area}(A)}{\text{Area}(S)}.$$



Buffon's needle. 1777. a method to compute π.

$l < a$.

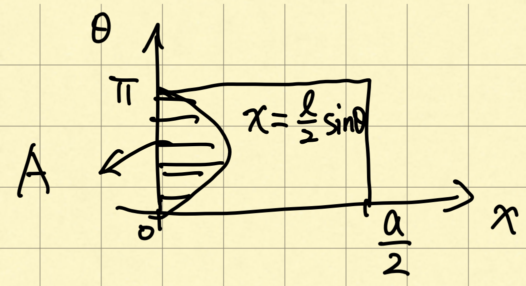


$$S = \{(x,\theta) \mid 0 \le \theta \le \pi,\ 0 \le x \le \tfrac{a}{2}\}.$$

A = needle intersects with a line.

$$\mathbb{P}(A) = ?$$

$$A = \{(x,\theta) \mid x \le \tfrac{l}{2}\sin\theta\}.$$



$$\text{Area}(S) = \pi \frac{a}{2}$$

$$\text{Area}(A) = \int_0^{\pi} \frac{l}{2} \sin\theta \, d\theta = l.$$

$$\mathbb{P}(A) = \frac{\text{Area}(A)}{\text{Area}(S)} = \frac{2l}{\pi a}.$$

$$\frac{n}{N} \approx \mathbb{P}(A) = \frac{2l}{\pi a}.$$

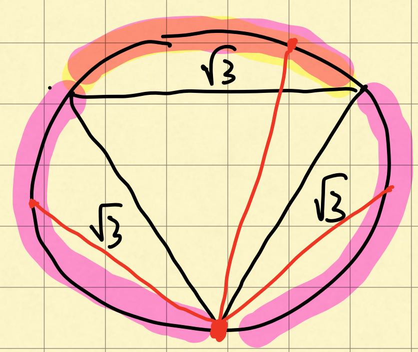
$$\pi \approx \frac{2lN}{na}.$$

<u>Monte · Carlo</u> Method.

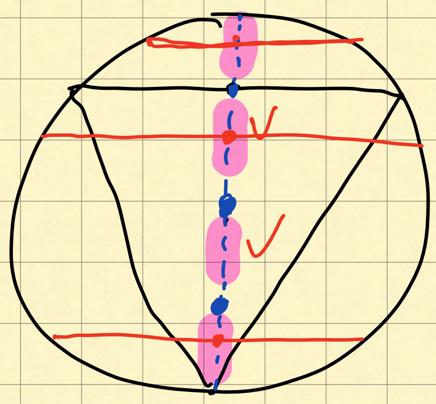
Bertrand's paradox.

Choose a chord of the unit circle at random. What is the probability that its length exceeds the side of the inscribe equilateral triangle. $(\sqrt{3})$.

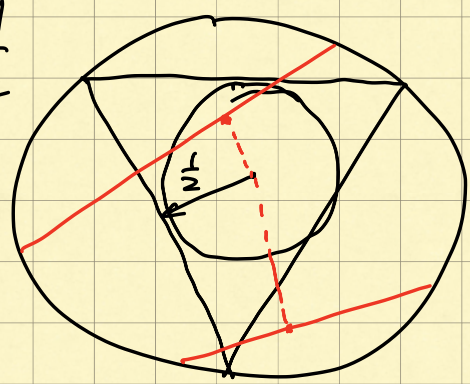
$$\frac{1}{3}$$



$\frac{2}{4} = \frac{1}{2}$.



$\frac{\pi \left(\frac{1}{2}\right)^2}{\pi \cdot 1^2} = \frac{1}{4}$



$\underline{\underline{S_1}}$

$\underline{\underline{S_2}}$

$\underline{\underline{S_3}}$

Classical $\mathbb{P}$ $\quad |S| < \infty \quad \mathbb{P}(w_i) = \mathbb{P}(w_j)$

Geometric $\mathbb{P}$ $\quad S$ nice shape "$\mathbb{P}(w_i) = \mathbb{P}(w_j)$"

- $\mathbb{P}(w_i) \neq \mathbb{P}(w_j)$
- countably infinite S.
- S doesn't nice shape.
- vagueness from verbal description

The modern definition.

$$\begin{array}{ccccc} \text{Experiment} & \Rightarrow & S = \text{outcomes} & \Rightarrow & \mathbb{P}(w_i). \\ & & \Downarrow & & \Downarrow \\ & & \text{define events.} & & \mathbb{P}(A). \\ & & \Downarrow & & \\ & & \mathbb{P}(A), \text{ where } A \text{ is already defined.} & & \end{array}$$

if $A \subset S$, then A is event ✓

S, defined which subsets of S are events.

Def (Event).

S. let $\mathcal{F}$ be a collection of subsets of S, such that:

(i) $S \in \mathcal{F}$.

(ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$.

(iii) $A_i \in \mathcal{F}$, $i = 1, 2, \ldots \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

Then $\mathcal{F}$ is called $\underbrace{\sigma\text{-algebra}}$ (event field).
elements in $\mathcal{F}$ are called events.

e.g. ① S. $\mathcal{F} = \{\phi, S\}$.

e.g. ② For a fix $A \subset S$

$$\mathcal{F} = \{\phi, A, A^c, S\}.$$

e.g. ③ $\underbrace{|S| < \infty.}$ $S = \{w_1, w_2, \dots, w_n\}$.

$$\mathcal{F} = \text{all subsets of } S = \mathcal{P}(S).$$

$$= \{\phi, \{w_1\}, \{w_2\}, \dots, \{w_n\}, \{w_1, w_2\}, \{w_1, w_3\}, \dots \dots \}$$

remark:

- $|S| < \infty$. or countably inf : $\mathcal{F} := \mathcal{P}(S)$ ✓
- S is uncountably inf : $\underbrace{\mathcal{F} := \mathcal{P}(S)}$ ×.

$\downarrow$

cannot define $\mathbb{P}$.

$(S, \underline{\mathcal{F}})$ no probability yet. $\mathbb{P}(w_i)$ gives a value.

Def (Probability). $\mathbb{P}(A) = \sum \mathbb{P}(w_i)$

$w_i \in A$

A probability is a non-negative function.

$\mathbb{P}$ defined on $\mathcal{F}$ such that

(i) $\mathbb{P}(S) = 1$

(ii) if $A_1, A_2, \dots, A_n, \dots \in \mathcal{F}$ are pairwise disjoint, then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

$(S, \mathcal{F}, \mathbb{P})$ is called probability space.

Properties of probability.

(i) $\mathbb{P}(\phi) = 0.$

$S = S \cup \phi \cup \phi \cup \phi \cdots$ disjoint union.

$$\mathbb{P}(S) = \mathbb{P}(S \cup \phi \cup \cdots)$$

$$= \mathbb{P}(S) + \mathbb{P}(\phi) + \cdots$$

$\mathbb{P}(S) = 1 \Rightarrow \mathbb{P}(\phi) = 0.$

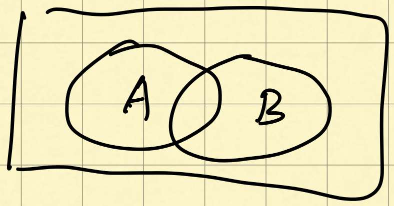
(ii) $A_1, \dots A_n$ are pairwise disjoint, then

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i)$$

(iii) $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$

(iv) $\mathbb{P}(A - B) = \mathbb{P}(A) - \mathbb{P}(A \cap B)$

L (v). $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.



$\mathbb{P}$ area