Lecture 6.

$\mathbb{P}$ set function defined on $\mathcal{F}$.

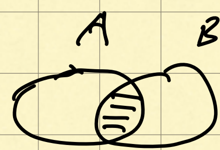
$(S, \mathcal{F}, \mathbb{P})$ is called probability space.

① $\mathbb{P}(\phi) = 0$

② $\mathbb{P}(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mathbb{P}(A_i)$, A_i pairwise disjoints

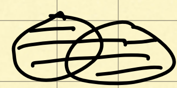
③ $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$

④ $\mathbb{P}(A - B) = \mathbb{P}(A) - \mathbb{P}(A \cap B)$



⑤ $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

$\mathbb{P}$ measures the "area" of events.



$\mathbb{P}(A \cup B)$.

$\forall A \in \mathcal{F}$. we can define its "area" $\mathbb{P}(A)$.

Measure Theory. $\mathbb{P}$

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e.g. $S = \{w_1, w_2, \ldots \ldots\}$.

$\mathcal{F} = \mathcal{P}(S)$

Define $\mathbb{P} : \mathcal{F} \longrightarrow [0, 1]$. as follows:

$\mathbb{P}(w_i) := a_i$ such that

$$\sum_{i=1}^{\infty} a_i = 1.$$

- $\mathbb{P}(S) = 1$
- $\mathbb{P}(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$

Experiment:
Toss a coin until the first head.

Head: q
Tail: $1-q$.

$$S = \{1, 2, 3, 4, \cdots\}.$$

$$\mathbb{P}(n) = (1-q)^{n-1} \cdot q \qquad \text{(Independence)}$$

$$\sum_{n=1}^{\infty} \mathbb{P}(n) = 1.$$

$$\mathbb{P}(\text{even number of tosses}) = \mathbb{P}(\{2, 4, 6, 8 \ldots\})$$

$$= \sum_{k=1}^{\infty} \mathbb{P}(2k) = \cdots$$

Conditional Probability & Independence

e.g. Toss a coin twice.

$$S = \{(H, H), (H, T), (T, H), (T, T)\}.$$

$$\mathbb{P}(2 \text{ heads}) = \frac{1}{4}.$$

There was at least one head.

$$\mathbb{P}(2 \text{ heads}) = \frac{1}{3}.$$

$S' = \{ (H, H), (H, T), (T, H) \}$

Def. (Conditional Probability). $(S, \mathcal{F}, \mathbb{P})$. and $B \in \mathcal{F}$. $\mathbb{P}(B) > 0$. The conditional probability of $\underline{A \text{ given } B}$ is

$$\mathbb{P}(A \mid B) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$ ✓

B is used to filter S. S' $\mathbb{P}(A)$

e.g. (Disease Testing)

Let A = random person has the disease.

$\mathbb{P}(A) = 1\%$. $\mathbb{P}(A^c) = 99\%$.

a test:

- <u>has the disease</u>, 99% the result is +
- no disease, 95% the result is −.

B = result is positive.

$$\mathbb{P}(B \mid A) = 0.99.$$
$$\mathbb{P}(B^c \mid A^c) = 0.95.$$

<u>Bayes' Theorem.</u>

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B \mid A) = \mathbb{P}(B)\mathbb{P}(A \mid B).$$

$$\mathbb{P}(A_1 \cap A_2 \cdots A_n) = \mathbb{P}(A_1)\mathbb{P}(A_2 \mid A_1)\mathbb{P}(A_3 \mid A_1 \cap A_2)$$

$\cdots \mathbb{P}(A_n | A_1 \cdots A_{n-1})$.

e.g. (Polya's urn)

r red balls

b black balls

repeatly: draw one ball, put it back, add c more balls of the same color.

Draw n times.

$\mathbb{P}$(the first n_1 times are black
the remain $n - n_1 := n_2$ times are red)

= ?

Let A_i = the i-th draw come out with the prescribed color.

A_1 = 1st is black

A_2 = 2nd is black

$\vdots$

A_{n_1+1} = n_1+1 is red.

$$\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_n)$$

$$\mathbb{P}(A_1) = \frac{b}{b+r}$$

$$\mathbb{P}(A_2 | A_1) = \frac{b+c}{b+r+c}$$

$$\mathbb{P}(A_3 \mid A_1 A_2) = \frac{b+2c}{b+r+2c}$$

$$\vdots$$

$$\mathbb{P}(A_{n_1} \mid A_1 \cdots A_{n_1-1}) = \frac{b+(n_1-1)c}{b+r+(n_1-1)c}$$

$$\vdots$$

$$\mathbb{P}(A_{n_1+1} \mid A_1 \cdots A_{n_1}) = \frac{r}{b+r+n_1c}$$

$$\vdots \qquad = \frac{r+c}{b+r+(n_1+c)c}$$

$$\vdots$$

$$\mathbb{P}(A_n \mid \cdots) = \frac{r+(n_2-1)c}{b+r+(n-1)c}.$$

$\mathbb{P}(A \mid B)$ $\quad$ $\mathbb{P}(A)$. $\quad$ which is larger ?

$A = \{\text{card is red}\}$. $\quad \mathbb{P}(A) = \frac{1}{2}$.

B	$\mathbb{P}(A \mid B)$
card is heart	1
card is spade	0
card is Ace	$\frac{1}{2}$

A, B are independent.

Def (Independence of two events).

A and B are independent if $\qquad A \perp\!\!\!\perp B.$

$$\mathbb{P}(A|B) = \mathbb{P}(A).$$

$$\Leftrightarrow \mathbb{P}(B|A) = \mathbb{P}(B).$$

$$\Leftrightarrow \boxed{\mathbb{P}(A\cap B) = \mathbb{P}(A)\mathbb{P}(B).}$$

$$\mathbb{P}(A\cap B) = \mathbb{P}(A)\overset{\mathbb{P}(B).}{\mathbb{P}(B|A)}$$

if $A \perp\!\!\!\perp B$ then $A^c \perp\!\!\!\perp B$, $A \perp\!\!\!\perp B^c$, $A^c \perp\!\!\!\perp B^c$.

remark. Independent $\neq$ mutually exclusive.

$\mathbb{P}(A\cap B) = \mathbb{P}(A)\mathbb{P}(B)$ $\qquad\qquad$ $A\cap B = \phi.$

$$\mathbb{P}(B|A) = 0.$$

A, B, C

$A_1, A_2, \ldots, A_n$

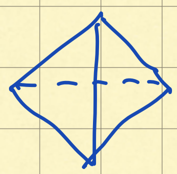
- pairwise independence. $\qquad \forall i, j \; i \neq j$: $\mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i)\mathbb{P}(A_j).$
- mutually independence:

$$\forall \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}.$$

$$\mathbb{P}(A_{i_1} \cap \cdots \cap A_{i_k}) = \mathbb{P}(A_{i_1}) \cdots \mathbb{P}(A_{i_k}).$$

Mutually independence $\Rightarrow$ pairwise independence.

e.g. (Bernstein's tetrahedron).



4 - faced die.

red
white
black
red, white, black.

$$\mathbb{P}(R) = \mathbb{P}(W) = \mathbb{P}(B) = \frac{2}{4} = \frac{1}{2}.$$

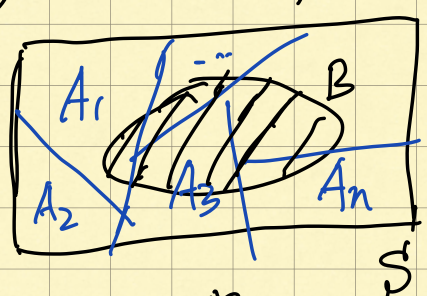
$$\mathbb{P}(R\cap W) = \mathbb{P}(R\cap B) = \mathbb{P}(W\cap B) = \frac{1}{4}. = \frac{1}{2}\times\frac{1}{2}.$$

$$\underbrace{\mathbb{P}(R\cap W\cap B)} = \frac{1}{4}. \neq \mathbb{P}(R)\,\mathbb{P}(W)\,\mathbb{P}(B) = \frac{1}{8}$$

pairwise independent only. .

Law of total probability.

$\{A_i\}$ is a partition for S.



$$\mathbb{P}(B) = \sum_{i=1}^{\infty} \mathbb{P}(B\cap A_i)$$

$$= \sum_{i=1}^{\infty} \mathbb{P}(A_i)\,\mathbb{P}(B \mid A_i)$$