Lecture 7.

$\{A_i\}_{i=1}^{\infty}$ is a partition for $S$.

$$\mathbb{P}(B) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)\,\mathbb{P}(B \mid A_i).$$

Example. Lost umbrella.

$\mathbb{P}(A_1) = 0.5$ library

$\mathbb{P}(A_2) = 0.3$ classroom.

$\mathbb{P}(A_3) = 0.2$ bus.

$B =$ found.

$\mathbb{P}(B \mid A_1) = 0.8$

$\mathbb{P}(B \mid A_2) = 0.6$

$\mathbb{P}(B \mid A_3) = 0.05$

$$\mathbb{P}(B) = \sum \mathbb{P}(A_i)\,\mathbb{P}(B \mid A_i) = 0.59.$$

Bayes Theorem

Let $\{A_i\}$ be a partition of $S$. let $\mathbb{P}(B) > 0$.

$$\mathbb{P}(A_i \mid B) = \frac{\mathbb{P}(A_i)\,\mathbb{P}(B \mid A_i)}{\sum \mathbb{P}(A_j)\,\mathbb{P}(B \mid A_j)}$$

$$\sum_i \mathbb{P}(A_i)\,\mathbb{P}(B|A_i)\,)$$

$$\mathbb{P}(A_1|B) = \frac{\mathbb{P}(A_1)\mathbb{P}(B|A_1)}{\mathbb{P}(A_1)\mathbb{P}(B|A_1) + \mathbb{P}(A_2)\mathbb{P}(B|A_2) + \mathbb{P}(A_3)\mathbb{P}(B|A_3)}$$

$$= \frac{0.5 \cdot 0.8}{0.5 \cdot 0.8 + 0.3 \cdot 0.6 + 0.2 \cdot 0.05} = \underline{0.678\ldots}$$

$$\mathbb{P}(A_2|B) = \underline{0.305}.$$

$$\mathbb{P}(A_3|B) = \underline{0.017\ldots}$$

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Proof: $$\mathbb{P}(A_i|B) = \frac{\mathbb{P}(A_i \cap B)}{\mathbb{P}(B)}$$

$$\overset{\text{L.o.T. to } \mathbb{P}(B)}{=} \frac{\mathbb{P}(A_i \cap B)}{\sum_i \mathbb{P}(A_i)\,\mathbb{P}(B|A_i)}$$

$$= \frac{\mathbb{P}(A_i)\,\mathbb{P}(B|A_i)}{\sum_i \mathbb{P}(A_i)\,\mathbb{P}(B|A_i)}.$$

| | | |
|---|---|---|
| $\mathbb{P}(A_i)$ | Prior | (knowledge) given. |
| $\mathbb{P}(B\|A_i)$ | likelihood | how well cause $i$ explains the consequence. |
| $\mathbb{P}(A_i\|B)$ | posterior | updated belief. |

$$\mathbb{P}(A_3) = 0.2 = 20\%$$

$$\mathbb{P}(A_3|B) = 0.017 < 2\%.$$

$A =$ fair coin.

coin $\quad \mathbb{P}(A) = \underline{\underline{0.5}}.$ $\quad A^c =$ only has head

Toss it $n$ times and get all heads. $= B$.

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A)\mathbb{P}(B \mid A)}{\mathbb{P}(A)\mathbb{P}(B \mid A) + \mathbb{P}(A^c)\mathbb{P}(B \mid A^c)}$$

$$= \frac{0.5 \cdot (\frac{1}{2})^n}{0.5 \cdot (\frac{1}{2})^n + 0.5 \cdot 1} \to 0.$$

as $n \to \infty$.

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$A =$ person has disease.

$\boxed{\mathbb{P}(A) = 1\%}$ $\quad$ $\underline{\underline{\text{rare}}}$

$\pm \left\{ \begin{array}{l} A \quad 99\% \ \ 1\% \quad c \\ 5\% \quad c \end{array} \right.$

$\mathbb{P}(+ \mid A) = \underline{\underline{0.99}}.$

$\mathbb{P}(- \mid A^c) = 0.95.$

$$\mathbb{P}(A \mid +) = \frac{\mathbb{P}(A)\mathbb{P}(+ \mid A)}{\mathbb{P}(A)\mathbb{P}(+ \mid A) + \mathbb{P}(A^c)\mathbb{P}(+ \mid A^c)}$$

$$= \frac{1\% \cdot 0.99}{1\% \cdot 0.99 + 99\%\,(1 - 0.95)}$$

$$= \frac{1}{6} \approx 16.7\%.$$

$A$ = person has disease.

$\mathbb{P}(A) = \cancel{1\%}$ $99\%$ common.

$\mathbb{P}(+ \mid A) = \underline{\underline{0.99}}$.

$\mathbb{P}(- \mid A^c) = \underline{\underline{0.95}}$.

$$\mathbb{P}(A \mid +) = \frac{\mathbb{P}(A)\,\mathbb{P}(+ \mid A)}{\mathbb{P}(A)\,\mathbb{P}(+ \mid A) + \mathbb{P}(A^c)\,\mathbb{P}(+ \mid A^c)}$$

$$= \frac{\cancel{1\%}\ 99\% \cdot 0.99}{\cancel{1\%}\ 99\% \cdot 0.99 + \cancel{99\%}\ 1\%\ (1 - 0.95)}$$

$$= \frac{0.9801}{0.9806} \approx \boxed{99\%}$$

$16.7\%$

$\mathbb{P}(B \mid A)$ is sensitive to $\mathbb{P}(A)$

$\mathbb{P}(B \mid A) \propto \mathbb{P}(A)$.

① Increase testing numbers.

$+_1$ $+_2$

$\mathbb{P}(A \mid +_1 \cap +_2) = 80\%$ v.s. $16.7\%$

② $\underline{\mathbb{P}(A) \text{ is low.}}$ Increase the prior $\mathbb{P}(A)$.

Conduct the test to risky people.

③ Test $\mathbb{P}(- \mid A^c)$.

Random Variables. (function).

$(S, \mathcal{F}, \mathbb{P})$. no functions yet.

Ex. $S = \underline{\{\overset{\nearrow 0}{\text{heart}}, \overset{\nearrow 0}{\text{diamond}}, \overset{\nearrow 1}{\text{spade}}, \overset{\nearrow 1}{\text{club}}\}}$.  $X$: heart $\to 0$, diamond $\to 0$, spade $\to 1$, club $\to 1$

red / black.
$0 \quad 1$  $\simeq \{0, 1\}$.

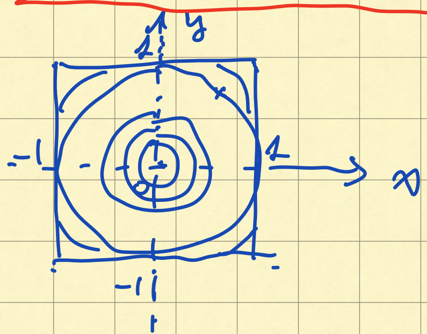
$n$ games.

$S = \{\overset{\nearrow n}{WWW\cdots W}, \overset{\nearrow 0}{LL\cdots L}, \overset{\nearrow 3}{WLWLLL\cdots}, \ldots\}$

$|S| = 2^n$.  $X(WWL\cdots LW) :=$ number of $W$.

$-1 \quad +1$  number of wins.

$\underline{\{0, 1, 2, 3, \ldots, n\}}$.  $X((x,y)) := \sqrt{x^2+y^2}$.



$S = \left\{(x,y) \,\middle|\, \begin{array}{l} |x| \le 1 \\ |y| \le 1 \end{array}\right\}$.

$\underline{[0, \sqrt{2}]}$.

Random variable

Let $(S, \mathcal{F}, \mathbb{P})$ be a probability space.

A function $X$ (or $\xi$) : $S \longrightarrow \mathbb{R}$ is a random variable. (if $\forall x \in \mathbb{R}$ $\underline{\{w : X(w) \leq x\}} \in \mathcal{F}$).

- numeric.
- reduced the size of $S$.