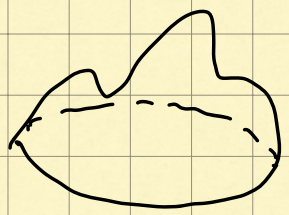
Lecture 12. Continuous Random Variables.

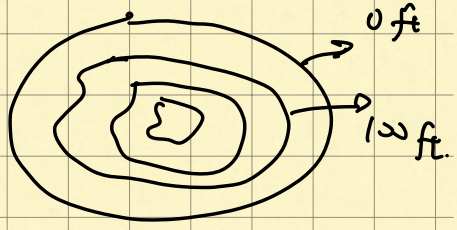
$\xi : S \longrightarrow \mathbb{R}$. is continuous when ξ is not discrete. i.e. values of ξ can fill in intervals.

Height of a mountain.



$\xi \in [0\text{ ft}, 10{,}000\text{ ft}]$.

$\mathbb{P}\{\xi = 3{,}000\text{ ft}\} = 0.$

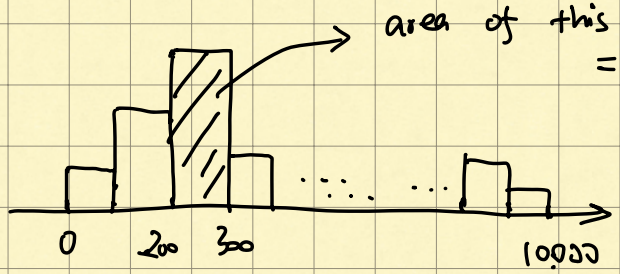


$\mathbb{P}\{\xi = k\} > 0$ for discrete r.v.

$\forall c,\ \mathbb{P}\{\xi = c\} = 0$ for continuous r.v.

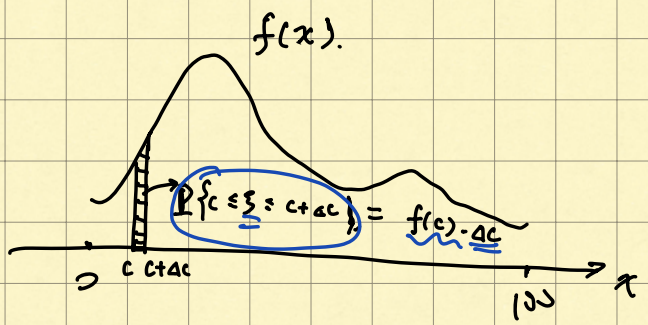
No pmf for continuous r.v.

$[0, 100\text{ ft}]$, $[100\text{ ft}, 200\text{ ft}]$, ... $[9{,}900\text{ ft}, 10{,}000\text{ ft}]$



area of this bar = height of this bar $\times$ 100 = probability of $\{200 \le \xi \le 300\}$.

Now Shrink the width of each interval.



This $f(x)$ is the probability density function (pdf) of ξ.

(Def pdf) The pdf of ξ is a function $f(x)$ such that $\forall\ a < b$

$$\mathbb{P}\{a \le \xi \le b\} = \int_a^b f(x)\,dx.$$

properties of pdf: · $f(x) \ge 0$

· $\int_{-\infty}^{+\infty} f(x)\,dx = 1.$

$$\mathbb{P}\{a \le \xi \le b\} = \mathbb{P}\{a < \xi \le b\} + \mathbb{P}\{\xi = a\}.$$

$$\mathbb{P}\{a \le \xi \le b\} = \mathbb{P}\{a \le \xi < b\} = \mathbb{P}\{a < \xi < b\} = \mathbb{P}\{a < \xi \le b\}.$$

(The crucial warning)

$$f(x) \ne \mathbb{P}\{\xi = x\}.$$

$$f(x) > 1$$

$f(c)$ is large $\Leftrightarrow$ $\mathbb{P}\{c-\delta \le \xi \le c+\delta\}$ is large.

Def (cumulative distribution function, cdf).

$$F_\xi(x) = \mathbb{P}\{\xi \le x\} = \int_{-\infty}^{x} f(t)\,dt$$

Therefore, when f is continuous,

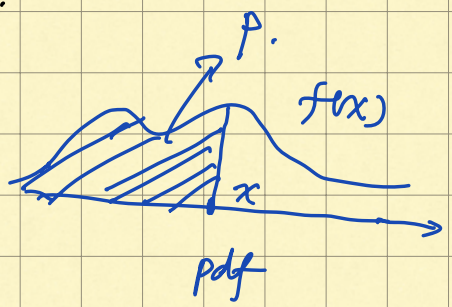
$$f(x) = F_\xi'(x).$$

$$\mathbb{P}\{a \ominus \xi \ominus b\} = F_\xi(b) - F_\xi(a)$$
$$\mathbb{P}\{\xi > a\} = 1 - F_\xi(a).$$

Def (percentile). For $0 < p < 1$, the $(100p)$-th percentile of ξ is the value x_p such that

$$\mathbb{P}\{\xi \le x_p\} = p.$$

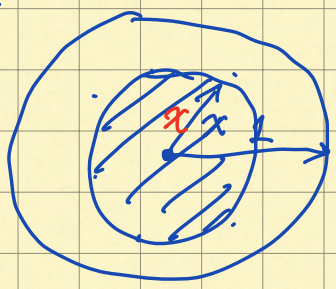
i.e. $\boxed{F_\xi(x_p) = p.}$



solve x for

$F_\xi(x) = p \Rightarrow x_p.$

e.g.



$\xi :=$ distance from the center.

$F_\xi(x) = \mathbb{P}\{\xi \le x\}$.

$= \frac{\pi x^2}{\pi \cdot 1^2} = x^2$.

$(0 \le x \le 1)$.

$f_\xi(x) = F_\xi'(x) = 2x$.

$E[\xi] = \sum_i x_i p(x_i)$

$\mu = E[\xi] = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$ $\quad dx$

center of mass.

$E(h(\xi)) = \int_{-\infty}^{\infty} h(x) f(x)\,dx$.

$\mathrm{Var}(\xi) = E[\xi^2] - E[\xi]^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$.

(Tail Formula) If $\xi \ge 0$ is continuous, then

$E[\xi] = \int_0^\infty \mathbb{P}\{\xi \ge x\}\,dx = \int_0^\infty (1 - F(x))\,dx$.

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e.g. uniform distribution on $[A, B]$.

$$f(x) = \begin{cases} \frac{1}{B-A} & x \in [A, B] \\ 0, & x \in [A, B) \end{cases}$$

cdf of uniform distribution.

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

$$= \frac{1}{B-A}\int_{A}^{x} 1\,dt$$

$$= \frac{x-A}{B-A}.$$

0 1

$A \le x \le B$.

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e.g. $U_1, U_2, \dots, U_n$ uniform distributions on $[0,1]$. independent. Find the expected value of the $\max(U_1, \dots, U_n)$ and $\min(U_1, \dots, U_n)$.

For the maximum. cdf of MAX.

$$\mathbb{P}\{MAX \le x\} = \mathbb{P}\{U_1 \le x, U_2 \le x, \dots, U_n \le x\}.$$

$$= \mathbb{P}\{U_1 \le x\}\,\mathbb{P}\{U_2 \le x\} \cdots \mathbb{P}\{U_n \le x\}.$$

$$= x^n$$

pdf of MAX is nx^{n-1}

$$\mathbb{E}(MAX) = \int_0^1 x \cdot nx^{n-1}\,dx = \frac{n}{n+1}.$$

For MIN ≥ 0.

$$\mathbb{P}\{MIN \ge x\} = \mathbb{P}\{U_1 \ge x, U_2 \ge x, \dots, U_n \ge x\}.$$

$$= \mathbb{P}\{U_1 \ge x\} \cdots \mathbb{P}\{U_n \ge x\}.$$

$$= (1 - \mathbb{P}\{U_1 < x\}) \cdots (1 - \mathbb{P}\{U_1 < x\})$$
$$= (1-x)^n .$$

$$\boxed{\mathbb{E}(MIN) = \int_0^1 (1-x)^n \, dx = \frac{1}{n+1}}$$

$$\mathbb{P}\{MIN \geq x\} = 1 - F_{MIN}(x) = (1-x)^n .$$

$$F_{MIN}(x) = 1 - (1-x)^n$$

$$f_{MIN}(x) = F'_{MIN}(x) = n(1-x)^{n-1} .$$

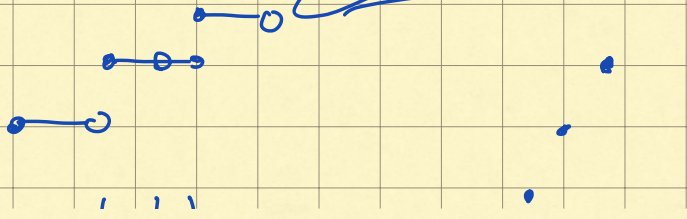
$$\mathbb{E}(MIN) = \int_0^1 x \cdot f_{MIN}(x)\, dx = \int_0^1 x \cdot n(1-x)^{n-1} dx$$
$$= \frac{1}{n+1} .$$

$$\mathbb{P}\{a \leq \xi \leq b\}$$
$$\| $$
$$\mathbb{P}\{a < \xi \leq b\} + \mathbb{P}\{\xi = a\}$$

ξ.

continuous
discrete · ξ.

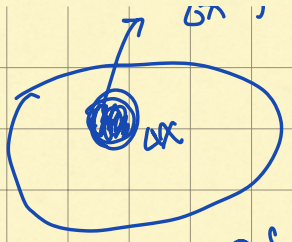
discrete.



a b c

a < b < c

P(X = b)



continuous

pdf

pdf