Lecture 13. Normal Distribution.

$X_1, X_2 \cdots X_N$ independent & identical distributions

$E(X_i) < \infty$.

$$Z := \frac{1}{N} \sum_{i=1}^{N} X_i .$$

Fact: when N is large enough,

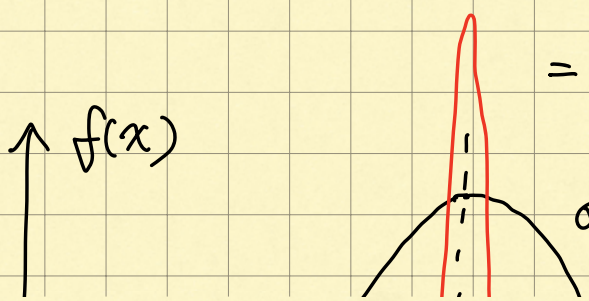
the distribution of Z "converges" to Normal Distribution:

Def Normal Distribution $\mathcal{N}(\mu, \sigma^2)$.

$$f(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in \mathbb{R}.$$

- $\int_{-\infty}^{\infty} f(x; \mu, \sigma)\, dx = 1.$
- $\xi \sim \mathcal{N}(\mu, \sigma^2)$. $E(\xi) = \int_{-\infty}^{\infty} x f(x; \mu, \sigma)\, dx = \mu.$
- $\xi \sim \mathcal{N}(\mu, \sigma^2)$, $Var(\xi) = E(\xi^2) - \mu^2$

$= \sigma^2$.





Def (Standard Normal Distribution).

$$\xi = N(0,1).$$

pdf $$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

cdf $$\Phi(x) = \mathbb{P}\{\xi \le x\} = \int_{-\infty}^{x} \varphi(t)\,dt = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,dt$$

(Φ-table).

x	$\Phi(x)$
0	0.5
0.2	0.6xx
⋮	⋮

(standardization). $\xi \sim N(\mu, \sigma^2)$.

Then $$\xi := \frac{\xi - \mu}{\sigma}.$$

$$\xi \sim N(0,1).$$

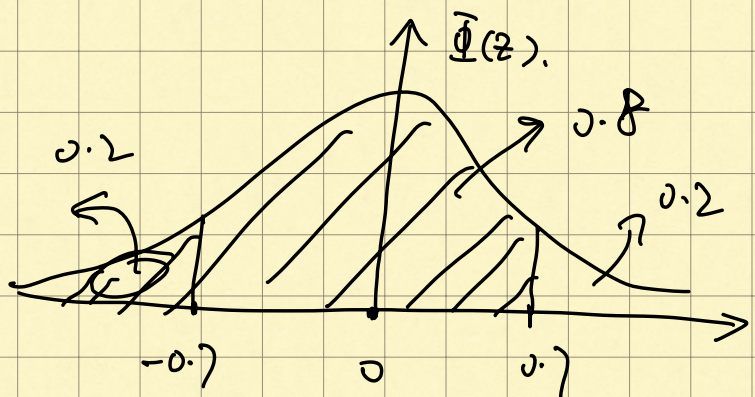
$\sim N(0,1)$

$$P\{\xi \le x\} = P\left\{\frac{\xi-\mu}{\sigma} \le \frac{x-\mu}{\sigma}\right\}.$$

$$= \Phi\left(\frac{x-\mu}{\sigma}\right). \quad = 0.895$$

$$\frac{x-\mu}{\sigma} = 0.728$$

z	$\Phi(z)$
⋮	⋮
$\frac{x-\mu}{\sigma}$	0.895
⋮	⋮



z	$\Phi(z)$
0	0.5
0.15	
0.20	
0.25	
⋮	⋮

$\Phi(0.7) = 0.8$

$\Phi(-0.7)$.

$$P\{a \le \xi < b\}.$$

$$= P\left\{\frac{a-\mu}{\sigma} \le Z < \frac{b-\mu}{\sigma}\right\}.$$

$$= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right).$$

$\xi \sim N(\mu, \sigma^2)$.

q_0-th percentile ?

Find x_{q_0} such that

$$P\{\xi \le x_{q_0}\} = q_0.$$

$$\Rightarrow P\left\{\frac{\xi-\mu}{\sigma} \le \frac{x_{q_0}-\mu}{\sigma}\right\}$$

$$\Rightarrow \Phi\left(\frac{x_{q_0}-\mu}{\sigma}\right)$$

Find x_{q_0} such that

$$\Phi\left(\frac{x_{q_0}-\mu}{\sigma}\right) = q_0.$$

z	$\Phi(z)$
⋮	⋮
2	0.9

$$\Phi(2) = 0.9$$

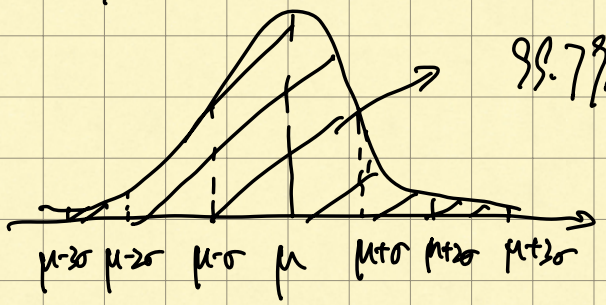
$$\frac{x_{q_0}-\mu}{\sigma} = 2.$$

The emiprical Rule $\xi \sim N(\mu, \sigma^2)$.

$$P\{\mu-\sigma \le \xi \le \mu+\sigma\} \approx 68\%$$

$$P\{\mu-2\sigma \le \xi \le \mu+2\sigma\} \approx 95\%$$

$$P\{\mu-3\sigma \le \xi \le \mu+3\sigma\} \approx 99.7\%$$



$\mu = 175$ cm

$\sigma^2 = 5$

160 – 180

165 – 185

$\xi \sim$ Binomial (n, p).

$\mu = E[\xi] = np$.

$\sigma^2 = Var(\xi) = np(1-p)$.

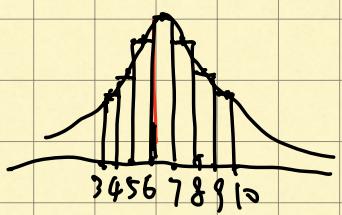
$n \gtrsim 30$

For large n:

$$P\{\xi \le x\} \approx \Phi\left(\frac{x + 0.5 - np}{\sqrt{np(1-p)}}\right)$$

→ continuity correction.

$$= \Phi\left(\frac{x + 0.5 - \mu}{\sigma}\right).$$



Change of Variable.

X is a c.r.v. f_X is the pdf.

g is a smooth fuction. monotone

$\boxed{Y = g(X)}$ is also a c.r.v.

What is f_Y pdf of Y ?

① cdf of Y:

$$F_Y(y) = \mathbb{P}\{Y \le y\}$$

$$= \mathbb{P}\{g(X) \le y\}.$$

$$= \begin{cases} \mathbb{P}\{X \le g^{-1}(y)\}, & g \text{ increasing.} \\ \mathbb{P}\{X \ge g^{-1}(y)\}. & g \text{ decreasing.} \end{cases}$$

$$\boxed{F_Y(y) = \begin{cases} F_X(g^{-1}(y)). & g \nearrow \\ 1 - F_X(g^{-1}(y)). & g \searrow \end{cases}}$$

② $f_Y(y) = F_Y'(y)$.

$$= \begin{cases} f_X(g^{-1}(y)) \cdot \frac{d}{dy}(g^{-1}(y)). & g \nearrow \\ - f_X(g^{-1}(y)) \cdot \frac{d}{dy}(g^{-1}(y)). & g \searrow. \end{cases}$$

$X \sim N(0,1)$. $Y = X^2 \geq 0$

$$F_Y(y) = \mathbb{P}\{X^2 \leq y\} = \mathbb{P}\{-\sqrt{y} \leq X \leq \sqrt{y}\}$$
$$= 2\Phi(\sqrt{y}) - 1.$$

$$f_Y(y) = 2 \cdot \varphi(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} = \frac{1}{\sqrt{2\pi y}} e^{-y/2}, \quad y > 0.$$

χ^2 distribution. 1 degree of freedom.