The exponential distribution.

Recall the Poisson process with rate $\lambda > 0$.

window of length t. λt

Poisson(λt). counts the number of arrivals during time window of length t.

$$\mathbb{P}\{\xi = k\} = \frac{(\lambda t)^k}{k!} e^{-\lambda t} \quad . \quad k = 0, 1, 2, \ldots$$

how long until the first arrival ?

X $\longrightarrow$ Exponential Distribution.

X be the waiting of the first arrival.

$$\mathbb{P}\{X \le t\} = \mathbb{P}\{\text{at least one arrival in } [0, t]\}.$$

$$= 1 - \mathbb{P}\{\text{no arrival in } [0, t]\}$$

$$= 1 - \mathbb{P}\{\xi = 0\}.$$

$$= 1 - \frac{(\lambda t)^0}{0!} e^{-\lambda t}$$

$$= 1 - e^{-\lambda t} \quad .$$

Def (Exponential distribution). $X \sim \text{Exp}(\lambda)$.

$$F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0. \\ 0, & x < 0. \end{cases}$$

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

$$E(X) = \frac{1}{\lambda}, \quad Var(X) = \frac{1}{\lambda^2}.$$

The memoryless property. $X \sim Exp(\lambda)$.

$$\mathbb{P}\{X \geq t_0 + t \mid X \geq t_0\} = \mathbb{P}\{X \geq t\} = e^{-\lambda t}.$$

pf: $\mathbb{P}\{X \geq t_0 + t \mid X \geq t_0\} = \dfrac{\mathbb{P}\{X \geq t_0 + t, X \geq t_0\}}{\mathbb{P}\{X \geq t_0\}}$

$$= \frac{\mathbb{P}\{X \geq t_0 + t\}}{\mathbb{P}\{X \geq t_0\}}$$

$$= \frac{1 - F(t_0 + t)}{1 - F(t_0)} = \frac{e^{-\lambda(t_0+t)}}{e^{-\lambda t_0}} = e^{-\lambda t}.$$

(Waiting for the bus). Buses arrive as a Poisson process with rate 6 per hour. You arrive at the stop at any time. How long do you expected to wait?

W is the waiting time.

$\mathbb{P}\{\text{Wait for at least } t \text{ hours}\} = \mathbb{P}\{W > t\}$

$= \mathbb{P}\{\text{no bus in the next } t \text{ hours}\}$

$= e^{-6t}$

$W \sim \text{Exp}(6)$. $\quad \mathbb{E}(W) = \frac{1}{6}\text{ hour} = 10 \text{ min}$.

Example. X = lifetime of a bulb.
Y = lifetime of another bulb.

$X \sim \text{Exp}(\frac{1}{1000})$ $\quad X \perp\!\!\!\perp Y$.

$Y \sim \text{Exp}(\frac{1}{500})$.

How long it takes until one bulb fails.

M = waiting time of one bulb fails.

$\mathbb{P}\{M > t\} = \mathbb{P}\{X > t\}\,\mathbb{P}\{Y > t\}$

$= e^{-\frac{t}{1000}} \cdot e^{-\frac{t}{500}}$

$= e^{-(\frac{1}{1000}+\frac{1}{500})t}$.

$M \sim \text{Exp}(\frac{1}{1000}+\frac{1}{500})$.

$\mathbb{E}(M) = \frac{1}{\frac{1}{1000}+\frac{1}{500}} \approx 333$ hours.

The Gamma Distribution. big picture..

The waiting time of first arrival $= \text{Exp}(\lambda)$.

The waiting time of the α-th arrival $=$ Gamma(α, β).

Definition (Gamma Function).

For $\alpha > 0$,

$$\Gamma(\alpha) := \int_0^\infty x^{\alpha-1} e^{-x}\, dx.$$

Proposition.

- $\Gamma(\alpha) = (\alpha-1)\,\Gamma(\alpha-1)$ For $\alpha > 1$.
- $\Gamma(n) = (n-1)!$ for positive integers n.

Definition (Gamma Distribution): $X \sim \text{Gamma}(\alpha, \beta)$

$$f(x;\alpha,\beta) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

The case $\beta = 1$ is the standard Gamma distribution $f(x;\alpha,1) = \frac{x^{\alpha-1}e^{-x}}{\Gamma(\alpha)}$.

How to read the α and β.

Take $\alpha = 1$, $\beta = \frac{1}{\lambda}$ = Expected value of waiting time of first arrival in the Poisson process.

$$f(x;1,\tfrac{1}{\lambda}) = \frac{1}{\frac{1}{\lambda}\Gamma(1)} \cdot x^0 e^{-\lambda x} = \underbrace{\lambda e^{-\lambda x}}_{\text{pdf of } \exp(\lambda)}.$$

$X \sim \text{Gamma}(\alpha, \beta)$. $\xrightarrow{\frac{1}{\beta} = \lambda}$ the rate in the Poisson process

- X: total waiting time
- α: α-th arrivals.

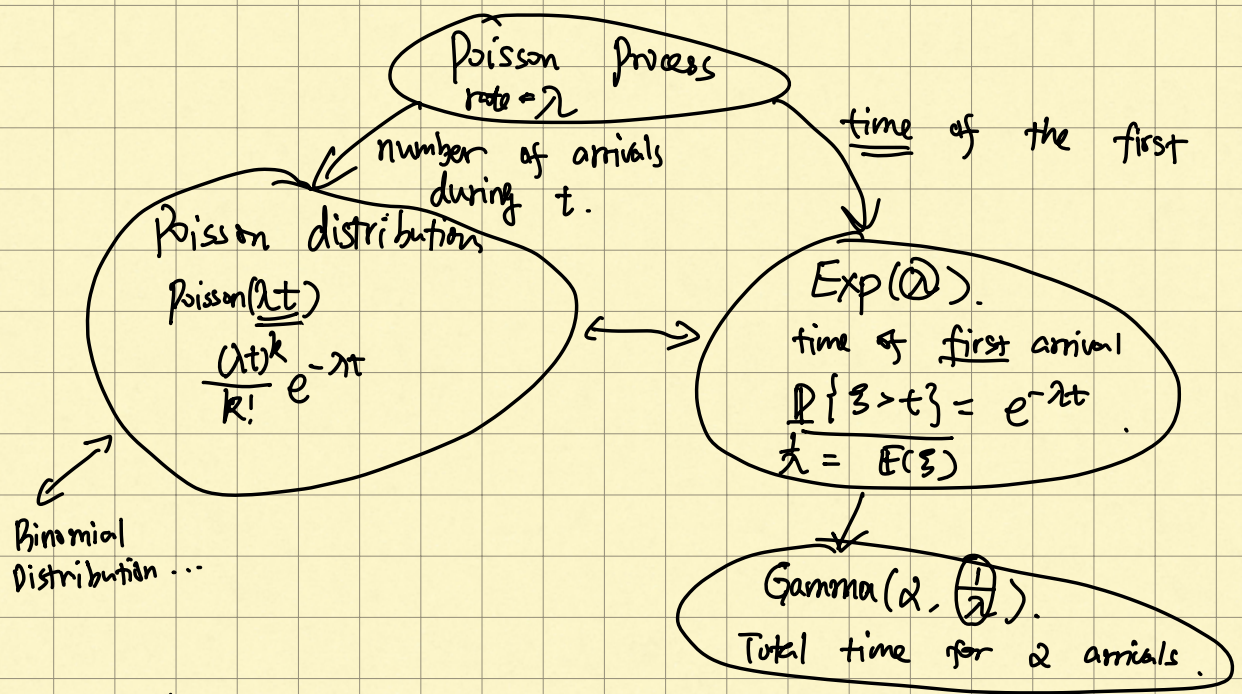
$\mathbb{E}(X) = 2 \cdot \beta$

$\mathrm{Var}(X) = 2 \cdot \beta^2$

β = expected waiting time for each arrival.

$X_1, X_2, \ldots, X_n \sim \mathrm{Exp}(\lambda)$.

$\sum_i X_i \sim \mathrm{Gamma}(n, \frac{1}{\lambda})$



(Scaling of Gamma distribution)

$X \sim \mathrm{Gamma}(2, \beta)$. $\frac{X}{\beta} \sim \mathrm{Gamma}(2, 1)$.

$f_X(x)$, $Y := \frac{X}{\beta}$. pdf of Y ?

$f_Y(y) \overset{(*)}{=} \beta \cdot f_X(\beta y) = \frac{y^{2-1} e^{-y}}{\Gamma(2)}$, pdf of