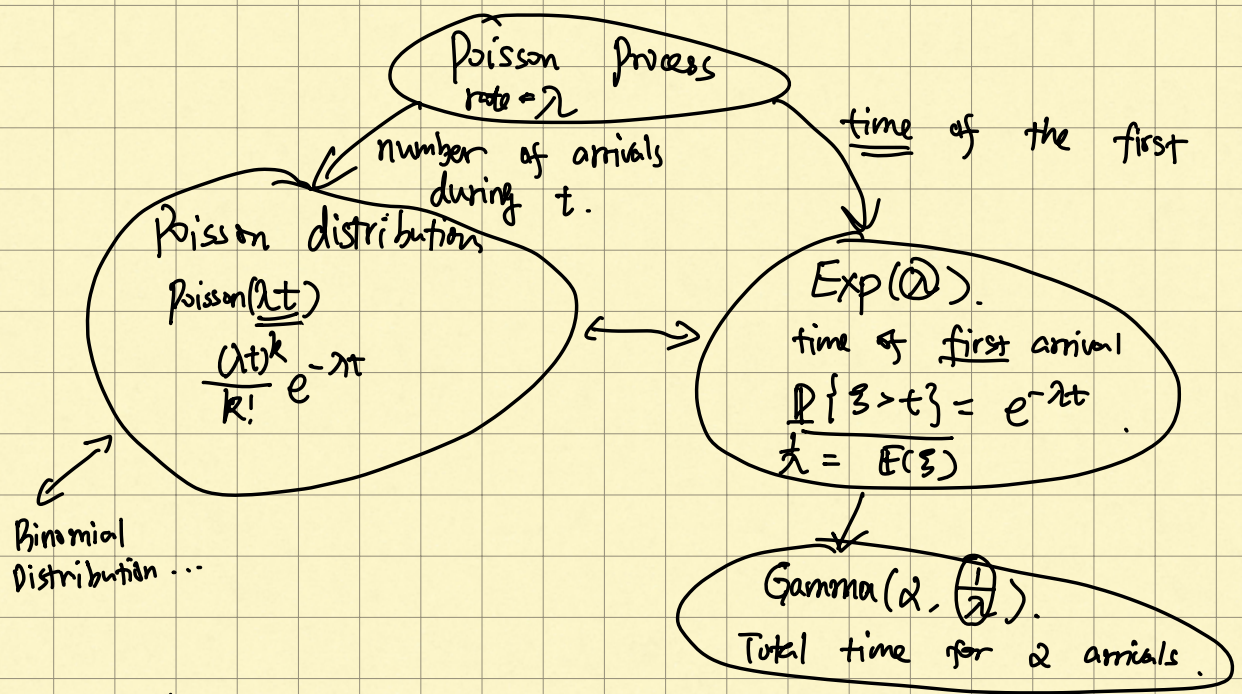
$\mathbb{E}(X) = 2 \cdot \beta$

$\mathrm{Var}(X) = 2 \cdot \beta^2$

β = expected waiting time for each arrival.

$X_1, X_2, \ldots, X_n \sim \mathrm{Exp}(\lambda)$.

$\sum_i X_i \sim \mathrm{Gamma}(n, \frac{1}{\lambda})$



(Scaling of Gamma distribution)

$X \sim \mathrm{Gamma}(2, \beta)$. $\frac{X}{\beta} \sim \mathrm{Gamma}(2, 1)$.

$f_X(x)$, $Y := \frac{X}{\beta}$. pdf of Y ?

$f_Y(y) \overset{(*)}{=} \beta \cdot f_X(\beta y) = \frac{y^{2-1} e^{-y}}{\Gamma(2)}$, pdf of

Gamma(2,1)

(Additivity). $X_1 \sim \text{Gamma}(\alpha_1, \beta_1)$

$X_2 \sim \text{Gamma}(\alpha_2, \beta_2)$.

$X_1 \perp\!\!\!\perp X_2$

$$\frac{X_1}{\beta_1} + \frac{X_2}{\beta_2} \sim \text{Gamma}(\alpha_1 + \alpha_2, 1).$$

Remark.

$$\text{Gamma}(3.14, \beta) = \text{Gamma}(1, \beta) + \text{Gamma}(2, \beta) + \text{Gamma}(0.14, \beta).$$

$\subset 1$.

- Chi-square
- Weibull
- lognormal
- Beta

Chi-square.

identical, independent, distributed from.

$Z_1, Z_2, \ldots, Z_k \overset{i.i.d.}{\sim} N(0,1)$.

$$Y := \sum_{i=1}^{k} Z_i^2$$

$Y \sim \chi_k^2$, k is the degree of freedoms.

$$f_Y(y;k) = \begin{cases} \frac{1}{2^{k/2}\Gamma(k/2)} y^{k/2-1} e^{-y/2}, & y > 0 \\ 0, & y \le 0. \end{cases}$$

$\chi_k^2 \ (=) \ \text{Gamma}(\frac{k}{2}, 2)$

Intuition of χ_k^2:

velocity $\vec{V} = (V_x, V_y, V_z)$ $\quad V_x, V_y, V_z \overset{iid}{\sim} N(0,1)$.

Kinetic energy $= \frac{1}{2} \cdot m \cdot |\vec{V}|^2$

$= \frac{1}{2} m \cdot (V_x^2 + V_y^2 + V_z^2)$

$= \frac{1}{2} m \cdot \chi_3^2$.

Weibull Distribution.

$X \sim \text{Weibull}(\alpha, \beta)$ $\alpha, \beta > 0$.

$$f_X(x;\alpha,\beta) = \begin{cases} \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-(x/\beta)^{\alpha}} & x>0 \\ 0 & x \le 0 \end{cases}.$$

$$F_X(x) = 1 - e^{-(x/\beta)^{\alpha}} \quad x \ge 0.$$

$$f_{\text{Gamma}}(x;\alpha,\beta) = \frac{1}{\beta\Gamma(\alpha)}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-(\frac{x}{\beta})^{1}}$$

Theorem. (Interpretation of Weibull).

$T \sim \text{Exp}(1)$ (rate λ), $X := \beta \cdot T^{1/\alpha}$.

$X \sim \text{Weibull}(\alpha, \beta)$.

No memoryless property.

$$\mathbb{P}\{X \ge t_0 + t \mid X \ge t_0\} \neq \mathbb{P}\{X \ge t\}.$$

Lognormal Distribution.

$\log$ $\ln = \log_e$

$\log = \ln = \log_e$

$\log X \sim N(\mu, \sigma^2)$

Then $X \sim \text{Lognormal}(\mu, \sigma^2)$.

$$F_X(x) = \mathbb{P}\{X \le x\} = \mathbb{P}\{\log X \le \log x\}$$

$$= P\left\{ \frac{\log X - \mu}{\sigma} \le \frac{\log x - \mu}{\sigma} \right\}$$

$$= \Phi\left(\frac{\log x - \mu}{\sigma} \right).$$

$$f_X(x) = \Phi'\left(\frac{\log x - \mu}{\sigma} \right) \cdot \frac{1}{\sigma x}$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \cdot e^{-\frac{(\log x - \mu)^2}{2\sigma^2}} \cdot \frac{1}{x}, \quad x > 0.$$

X stock price.

R_i iid unknown σ

$$X = X_0 (1+R_1)(1+R_2) \cdots (1+R_n).$$

$$\log X = \log X_0 + \sum_{i=1}^{n} \log(1+R_i) \overset{\text{approx}}{\sim} N(\mu, \sigma^2). \quad n \gg \text{large}.$$

$$\log X = N(\mu + \log X_0, \sigma^2).$$

$$X \sim \text{lognormal}(\mu + \log X_0, \sigma^2).$$

Beta Distribution (has bounded support).

Def (Standard Beta Distribution).

$X \sim$ Standard Beta(α, β). if

$$f(x; \alpha, \beta) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot x^{\alpha-1}(1-x)^{\beta-1}, & 0 \le x \le 1 \\ 0, & \text{elsewhere.} \end{cases}$$

The general Beta Distribution. is a Beta(α, β)

on a general interval $[A, B]$.

$$A \le Y \le B$$

$$0 \le \frac{Y-A}{B-A} \le 1$$

If $\frac{Y-A}{B-A} \sim$ Standard Beta (α, β)

$\Updownarrow$

$Y \sim$ Beta (α, β) on $[A, B]$.

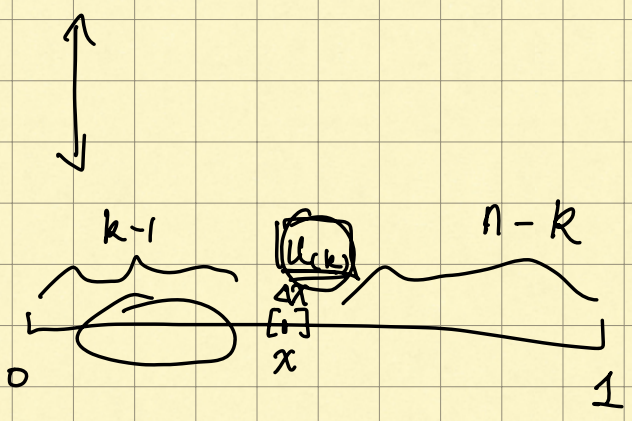
Theorem. $Y_1 \sim$ Gamma $(\alpha, 1)$. $Y_2 \sim$ Gamma $(\beta, 1)$.

$$X := \frac{Y_1}{Y_1 + Y_2} \sim \text{Beta}(\alpha, \beta).$$

Example. Drop n points uniformly on $[0, 1]$ independently. Let $U_{(k)}$ be the k-th smallest.

How is $U_{(k)}$ distributed?

Pf:



pdf · $n \cdot \binom{n-1}{k-1} x^{k-1} (1-x)^{n-k}$