Standard Beta.

$$f(x; \alpha, \beta) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, & 0 \le x \le 1 \\ 0 & \text{else.} \end{cases}$$

$X \in [0, 1]$

Y is a general Beta distribution on $[A, B]$

when $\frac{Y-A}{B-A} \sim$ Standard Beta(α, β)

$Y_1 \sim \text{Gamma}(\alpha, 1)$

$Y_2 \sim \text{Gamma}(\beta, 1)$

$$X = \frac{Y_1}{Y_1 + Y_2} \sim \text{Standard Beta}(\alpha, \beta).$$

$\in [0, 1]$.

Example. n iid uniform distributions on $[0, 1]$ $U_1, U_2, \cdots, U_n$.

Let $U_{(k)}$ be the k-th smallest.

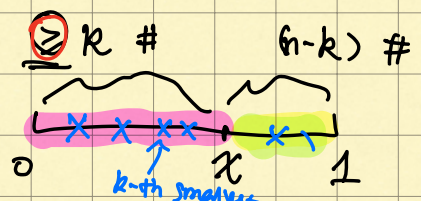
$$U_{(k)} = \min_{k\text{-th smallest}} (U_1, \cdots, U_n).$$

What is the distribution of $U_{(k)}$?

Approach 1: $F(x) = \mathbb{P}\{U_{(k)} \le x\}$.

$= \sum_{j=k}^{n} \binom{n}{j} x^j (1-x)^{n-j}$ ★

$\binom{n}{j}$ → # pts in $[0, x]$



x $\quad (1-x)$

j $\quad (n-j)$

$$f(x) = F'(x) = \sum_{j=k}^{n} \binom{n}{j} \left(j x^{j-1} (1-x)^{n-j} - x^j (n-j)(1-x)^{n-j-1} \right)$$

$j \ge k$

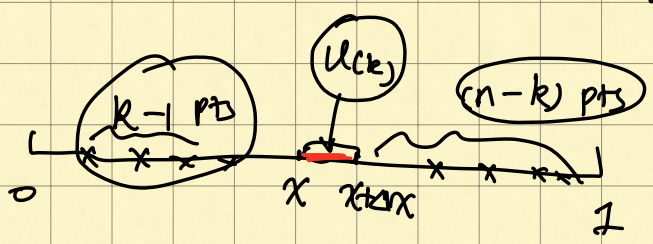
$$= \frac{n!}{(k-1)!\,(n-k)!} x^{k-1} (1-x)^{n-k}$$

pdf of Beta$(k, n-k+1)$.

Approach 2:

$f(x)\,dx = f(x)\,\Delta x$

$= \mathbb{P}\{ U_{(k)} \text{ lands in } [x, x+\Delta x] \}$.



$n-1$

$$= \underbrace{n \binom{n-1}{k-1} x^{k-1} (1-x)^{n-k}}\, dx.$$

pdf = pdf of Beta$(k, n-k+1)$

uniform $[A, B]$

$N(\mu, \sigma^2)$.

$\text{Exp}(\lambda) = \text{Gamma}(1, 1/\lambda)$

Gamma(α, β)

$\chi^2_k = \text{Gamma}(k/2, 2)$ $\qquad \chi^2_k = \sum_{i=1}^{k} N(0,1)^2$

Weibull (α, β) $\quad \beta T^{1/\alpha}$, $T \sim \text{Exp}(1)$.

Lognormal (μ, σ^2) if $\log(X) \sim N(\mu, \sigma^2)$.

Beta (α, β) $\quad \frac{Y_1}{Y_1+Y_2}$, Y_i gamma.

Joint Distributions.

$f(x)$. $\qquad X, Y$ → exam grades.

X: Study hours $\qquad$ X is not independent with Y.

$$\mathbb{P}\{X \le 2 \text{ and } Y \ge 90\} \neq \mathbb{P}\{X \le 2\}\,\mathbb{P}\{Y \ge 90\}.$$

joint probabilities.

joint pmf $\quad \mathbb{P}\{X = k, Y = m\}$.

joint pdf $\quad f(x, y)$

Discrete Case.

For discrete r.v.s X, Y, the joint pmf is

$$\mathbb{P}\{X = x, Y = y\} = p(x, y) \ge 0$$

$$\sum_{x}\sum_{y} \mathbb{P}\{X = x, Y = y\} = 1$$

$\sum_x \sum_y P(X=x, Y=y) = 1$.

$$P(A) = P\{(X,Y) \in A\} = \sum_{(x,y)\in A} p(x,y)$$

Example. (Insurance). You have 3 options for auto insurance: 100, 500, 1000.

3 options for home owner's deduction:

500, 1500, 5000.

X: the deduction of auto a random person selects.

Y: ——— home ———.

$p(x,y)$	$y=500$	$y=1500$	$y=5000$	$P_X(x)$
$x=100$	0.3	0.05	0	0.35
$x=500$	0.15	0.2	0.05	0.4
$x=1000$	0.1	0.1	0.05	0.25
$P_Y(y)$	0.55	0.35	0.10	1

joint pmf.

Def. Marginal pmf.

$$P_X(x) = \sum_y p(x,y), \qquad P_Y(y) = \sum_x p(x,y)$$

Continuous Case.

Def (Joint pdf) $f(x,y)$ is a joint pdf if $f(x,y) \geq 0$ and $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dxdy = 1$.

Then for any A.

$$\mathbb{P}(A) = \iint_{(x,y)\in A} f(x,y)\,dxdy$$

Example. $f(x,y) = \frac{6}{5}(x+y^2)$. $0 \le x \le 1$, $0 \le y \le 1$.

$$\mathbb{P}\left\{X \le \tfrac{1}{4},\ Y \le \tfrac{1}{4}\right\} = \int_0^{\frac{1}{4}}\int_0^{\frac{1}{4}} \frac{6}{5}(x+y^2)\,dxdy = \frac{7}{640}.$$

def (Marginal pdf)

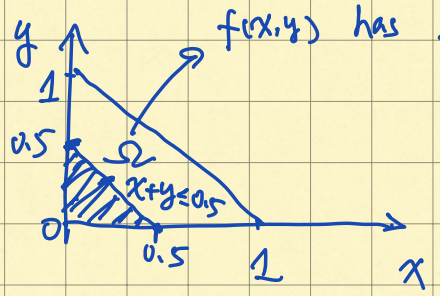
$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy, \qquad f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx.$$

Example. (Can of nuts) 1 - lbs. deluxed mixed of nuts.

X: weight of almonds
Y: weight of cashews
1-X-Y: weight of peanuts

$$f(x,y) = \begin{cases} 24xy, & 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le 1-x-y \le 1 \ (\Omega) \\ 0, & \text{else} \end{cases}$$

$$\mathbb{P}\{X+Y \le 0.5\} = \iint_{\substack{x+y\le 0.5 \\ (x,y)\in\Omega}} 24xy \, dx dy.$$



$$= \int_0^{\frac{1}{2}} \left(\int_0^{-x+0.5} 24xy \, dy \right) dx$$

$$= \int_0^{\frac{1}{2}} 12 \left(\tfrac{1}{2} - x\right)^2 dx$$

$$= \frac{1}{16} = 6.25\%.$$