Independence.

X, Y

discrete:

- $p(x, y) = \mathbb{P}\{X = x \text{ and } Y = y\}$.
- $p_X(x) = \sum_y p(x, y)$.
- $p_Y(y) = \sum_x p(x, y)$

if $X \perp\!\!\!\perp Y$, then $p(x, y) = p_X(x)\, p_Y(y)$.

continuous:

- $f(x, y) \not\equiv \mathbb{P}\{\ \ldots\ \}$
- $f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$.
- $f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$.

if $X \perp\!\!\!\perp Y$. then $f(x, y) = f_X(x) f_Y(y)$.

$X_1, \ldots, X_n$ independent continuous r.v.

$T_{max} = \max(X_1, \ldots, X_n)$

$T_{min} = \min(X_1, \ldots, X_n)$.

$F_{T_{max}}(t) =$

$\mathbb{P}\{T_{max} \le t\} = \mathbb{P}\{\max(X_1, \ldots, X_n) \le t\}$

$\mathbb{P}\{X_1 \le t, X_2 \le t, \ldots, X_n \le t\}$

$= \mathbb{P}\{X_1 \le t, X_2 \le t, \cdots, X_n \le t\}$

independence $\mathbb{P}\{X_1 \le t\}\,\mathbb{P}\{X_2 \le t\} \cdots \mathbb{P}\{X_n \le t\}$

$= F_{X_1}(t) \cdots F_{X_n}(t)$.

$$\int_{-\infty}^{t} \cdots \int_{-\infty}^{t} f(x_1, x_2, \cdots, x_n)\, dx_1 \cdots dx_n$$

$$= \int_{-\infty}^{t} \cdots \int_{-\infty}^{t} f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_n}(x_n)\, dx_1 \cdots dx_n$$

$$= \left(\int_{-\infty}^{t} f_{X_1}(x_1)\, dx_1\right) \cdots \left(\int_{-\infty}^{t} f_{X_n}(x_n)\, dx_n\right)$$

$T_{min} = \min(X_1, \cdots, X_n)$.

$F_{T_{min}}(t) =$

$\mathbb{P}\{T_{min} \le t\} = 1 - \mathbb{P}\{T_{min} > t\}$.

$\mathbb{P}\{T_{min} > t\} = \mathbb{P}\{\min(X_1, \cdots, X_n) > t\}$

$= \mathbb{P}\{X_1 > t, X_2 > t, \cdots, X_n > t\}$.

independence

$= \mathbb{P}\{X_1 > t\} \cdots \mathbb{P}\{X_n > t\}$

$= (1 - F_{X_1}(t)) \cdots (1 - F_{X_n}(t))$.

$$F_{T_{min}}(t) = 1 - (1 - F_{X_1}(t)) \cdots (1 - F_{X_n}(t)).$$

Conditional distributions.

X, Y $\quad f(x, y)$. $\rightleftharpoons$ $f_X(x)$ marginals, $f_Y(y)$.

Total weight = 1 lbs.

Can of nuts. $f(x, y) = 24xy$.

$0 < x < 1$
$0 < y < 1$
$0 < x + y < 1$

X: weight of almonds

Y: weight of Cashews

$1 - X - Y$: weight of peanuts.

Given 0.2 lbs of almonds. What is the distribution of weight of Cashews (Y)?

Def. Conditional pdf.

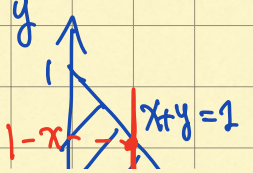
$$f_{Y|X}(y|x) := \frac{f(x, y)}{f_X(x)}.$$

$$\mathbb{P}(A|B) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

$$f_{X|Y}(x|y) := \frac{f(x, y)}{f_Y(y)}.$$

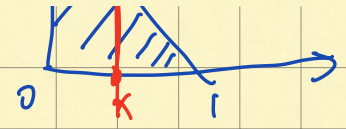
$$f_{Y|X}(y|x = 0.2) = \frac{24xy}{f_X(x)}\Big|_{x=0.2}$$

$$f_X(x) = \int_0^{1-x} 24xy\, dy = 12x(1-x)^2 \quad (0 < x < 1).$$



$$f_{Y|X}(y|x) = \frac{24xy}{12x(1-x)^2}$$

$0 < x < 1$



$= \frac{2y}{(1-x)^2}$, $0<y<1-x$.

$f_{Y|X}(y \mid 0.2) = \frac{2y}{0.64}$, $0<y<0.8$.

Expected Values.

For $h: \mathbb{R}^2 \to \mathbb{R}$, $(X, Y) \begin{cases} p(x,y) \\ f(x,y) \end{cases}$.

$$E(h(X,Y)) = \begin{cases} \sum_y \sum_x h(x,y)\, p(x,y) & \text{discrete} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x,y) f(x,y)\, dx\, dy & \text{continuous} \end{cases}$$

Example Five seats in a row.

X = seat you choose.

Y = seat the other person chooses.

1 2 3 4 5

Y \ X	1	2	3	4	5
1	0	$\frac{1}{20}$	$\frac{1}{20}$	×	·
2	$\frac{1}{20}$	0	×	·	×
3	·	·	0	×	·
4	$\frac{1}{20}$	$\frac{1}{20}$	·	0	×
5	·	·	·	·	0

$$p(x,y) = \begin{cases} \frac{1}{20}, & x, y \in \{1, \dots, 5\},\ x \neq y \\ 0, & \text{else.} \end{cases}$$

Expected

The number of seats between you?

$h(x,y) = |x-y| - 1.$ $\quad X = x$, $Y = y$.

~ ✓ — — —

$h(X,Y) = |X-Y| - 1$

$E(h) = ?$

$= \sum_y \sum_x h(x,y) p(x,y)$

$= \frac{1}{20} \sum_{x \neq y} (|x-y| - 1)$

$x \in \{1, \dots, 5\}$

$y \in \{1, \dots, 5\}.$

$= \frac{20}{20} = 1.$

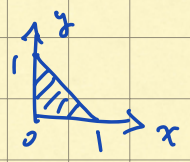
Example (cost of a can of nuts).

X	Almonds	\$1.5 /lb
Y	cashews	\$2.25 /lb
$1-X-Y$	peanuts	\$0.75/lb.

$h(X,Y) = 1.5X + 2.25Y + 0.75(1-X-Y)$

$= 0.75 + 0.75X + 1.5Y.$

$f(x,y) = 24xy.$



$$E(h(X,Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)\, f(x,y)\, dx dy$$

$$= \int_0^1 \int_0^{1-x} (0.75 + 0.75x + 1.5y)\, 24xy\, dy\, dx$$

$$= \$1.65.$$

$$E(h(X,Y)) = E(0.75 + 0.75X + 1.5Y)$$

$$= 0.75 + 0.75\, E(X) + 1.5\, E(Y)$$

$E(X)$: $\iint x\, 24xy\, dx dy$

$E(Y)$: $\iint y\, 24xy\, dx dy$.