$f(x, y)$. $\rightleftarrows$ $f_X(x)$, $f_Y(y)$. $\rightarrow$ $f_{X|Y}(x|y)$, $f_{Y|X}(y|x)$.

$$\mathbb{E}(h(X,Y)) = \iint h(x,y)\, f(x,y)\, dx\,dy.$$

$X|Y=y$ $\quad f_{X|Y}(x|y)$.

$$\mathbb{E}(X \mid Y=y) = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\, dx.$$

$=$ function of y.

Definition.

(Conditional Expectation.) ~~Expected Value~~

$$\mathbb{E}(h(X) \mid Y=y) = \int_{-\infty}^{\infty} h(x) \cdot f_{X|Y}(x|y)\, dx.$$

e.g. Can of Nuts.

$f(x,y) = 24xy$. $\quad 0 \le x \le 1$, $0 \le y \le 1$, $0 \le x+y \le 1$

$$f_{Y|X}(y|x) = \frac{2y}{(1-x)^2}, \quad 0 \le y \le 1-x.$$

$X = 0.2$ lbs of Almonds.

What is the Expected cost from peanuts?

$$\mathbb{E}\left((1-0.2-Y)\cdot 0.75 \mid X=0.2\right)$$

$$= \int_0^{0.8} 0.75(0.8-y) \cdot \frac{2y}{(1-0.2)^2}\, dy$$

$=$ ___

Followed-up question: if I want to control the cost of peanuts $\leq \$0.5$, then how much almonds I should put in the can?

$$\mathbb{E}\left((1-X-Y)\cdot 0.75 \mid X=x \right) \leq 0.5$$

$$\Rightarrow g(x) \leq 0.5$$

$\Rightarrow$ Solve x.

---

eg. Toss a fair coin until two consecutive tosses showing HH. What is the expected number of tosses?

$\mathbb{E}(N)$.

T	probability $\frac{1}{2}$	$\mathbb{E}(N \mid T) = 1 + \mathbb{E}(N)$.
HT	probability $\frac{1}{4}$	$\mathbb{E}(N \mid HT) = 2 + \mathbb{E}(N)$.
HH	probability $\frac{1}{4}$	$\mathbb{E}(N \mid HH) = 2$.

$$\mathbb{E}(N) = \frac{1}{2}(1+\mathbb{E}(N)) + \frac{1}{4}(2+\mathbb{E}(N)) + \frac{1}{4}(2) \quad \text{☆}$$

$$\Rightarrow \mathbb{E}(N) = \frac{3}{2} + \frac{3}{4}\mathbb{E}(N).$$

$$\Rightarrow \mathbb{E}(N) = 6.$$

Covariance.

X , $\mathrm{Var}(X) = \mathbb{E}\left((X-\mu_X)^2\right)$.

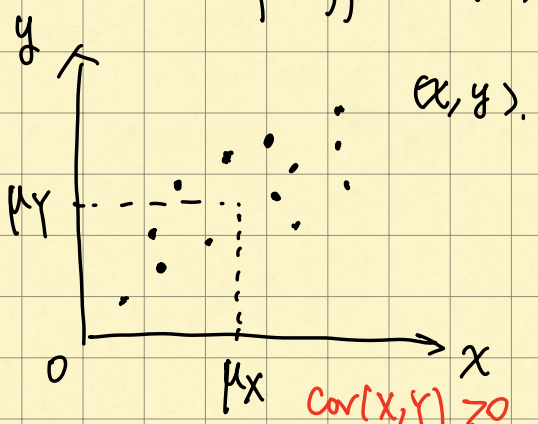
$\mu_X = \mathbb{E}(X)$.

X, Y. $p(x,y)$. $f(x,y)$

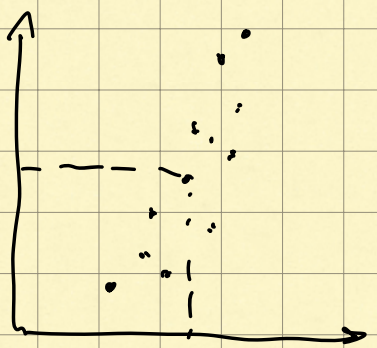
$\mathrm{Cov}(X,Y) := \mathbb{E}\left((X-\mu_X)(Y-\mu_Y)\right)$.

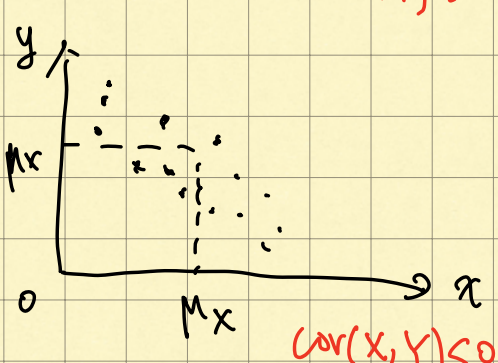
$= \sum_x \sum_y (x-\mu_X)(y-\mu_Y)\, p(x,y)$.

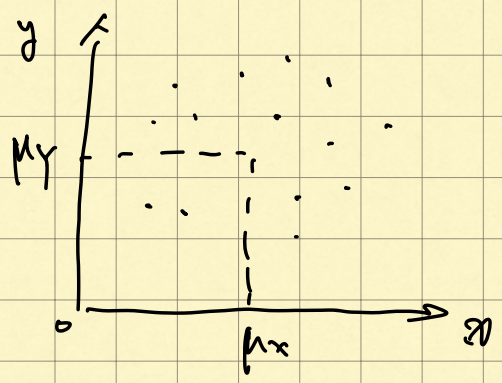
$\iint (x-\mu_X)(y-\mu_Y)\, f(x,y)\, dx\,dy$.



(x,y).







$\mathrm{Cov}(X,Y) = 0$.

X, Y. 若 $X = Y$

$\mathrm{Cov}(X,Y) = \mathrm{Cov}(X,X) = \mathrm{Var}(X)$.

$$\mathrm{Cov}(X,Y) = \mathbb{E}\big((X-\mu_X)(Y-\mu_Y)\big)$$
$$= \mathbb{E}\big((X-\mu_X)^2\big) = \mathrm{Var}(X).$$

- $X \perp\!\!\!\perp Y \Rightarrow \mathrm{Cov}(X,Y) = 0.$

The converse is false!

X, Y. $\mathrm{cov}(X,Y)=0$. but $X \not\perp\!\!\!\perp Y$.

$Y = X^2$.

$\mathrm{Cov}(X, X^2) =$

distribution of X

X	-1	0	1
P	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

$X \not\perp\!\!\!\perp Y$. ✓

Y	0	1
P	$\frac{1}{3}$	$\frac{2}{3}$

$$\mathrm{Cov}(X,Y) = \mathbb{E}\big((X-\mu_X)(Y-\mu_Y)\big)$$
$$= \mathbb{E}\big((X-0)(Y-\tfrac{2}{3})\big)$$
$$= \mathbb{E}\big(XY - \tfrac{2}{3}X\big).$$
$$= \mathbb{E}(XY) - \cancel{\tfrac{2}{3}\mathbb{E}(X)}$$
$$= \mathbb{E}(XY). \qquad Y = X^2.$$
$$= \mathbb{E}(X^3).$$
$$= -1\cdot\tfrac{1}{3} + 0 + 1\cdot\tfrac{1}{3}$$
$$= 0. \quad ✓$$

- Covariance measures the "linear" dependence between X and Y.
- $\mathrm{cov}(aX+b, cY+d) = ac\,\mathrm{Cov}(X,Y)$.

  $\mathrm{Var}(aX+b) = a^2\mathrm{Var}(X)$.
- $\mathrm{Cov}(X,Y) = \mathbb{E}(XY) - \mu_X \cdot \mu_Y$.

  (computational formula).

  $\mathrm{Var}(X) = \mathbb{E}(X^2) - \mu_X^2$.

$f(x,y) \rightarrow f_X(x), f_Y(y) \rightarrow \mu_X, \mu_Y \rightarrow \mathrm{Cov}(X,Y)$.

$f(x,y) \rightarrow \mathbb{E}(XY) = \iint xy f(x,y)\,dx\,dy \rightarrow \mathrm{Cov}(X,Y)$.

Can of Nuts. $f(x,y) = 24xy$ $\begin{pmatrix} 0 \le x+y \le 1 \\ 0 \le x \le 1 \\ 0 \le y \le 1 \end{pmatrix}$.

- $f_X(x) = 12x(1-x)^2$ $0 \le x \le 1$
- $f_Y(y) = 12y(1-y)^2$ $0 \le y \le 1$.

$\mu_X = \int_0^1 f_X(x)\,dx = \frac{2}{5}$

$\mu_Y = \int_0^1 f_Y(y)\,dy = \frac{2}{5}$.

$\mathbb{E}(XY) = \int_0^1 \int_0^{1-x} xy \cdot 24xy\,dx\,dy$

$= \frac{2}{15}$.



$\mathrm{Cov}(X,Y) = \frac{2}{15} - \left(\frac{2}{5}\right)\left(\frac{2}{5}\right) = -\frac{2}{75} < 0$.

$\mathrm{Cov}(X, Y)$

$\mathrm{Cov}(aX, bY) = ab\,\mathrm{cov}(X, Y)$.

X	weight in lbs	$16X$	oz.
Y	weight in lbs	$16Y$	oz.
	$\mathrm{cov}(X, Y)$ lbs.	$16^2\,\mathrm{cov}(X, Y)$	oz

Def (Correlation coefficient).

$$\mathrm{Corr}(X, Y) = \rho_{X,Y} := \frac{\mathrm{Cov}(X, Y)}{\sigma_X \sigma_Y}$$

$$= \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)} \cdot \sqrt{\mathrm{Var}(Y)}}.$$

$$= \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{cov}(X,X)} \cdot \sqrt{\mathrm{cov}(Y,Y)}}.$$

1. $\mathrm{Corr}(aX+b, cY+d) = \mathrm{corr}(X, Y)$.
2. $-1 \le \rho_{X,Y} \le 1$.
3. $\rho_{X,Y} = \pm 1$ if and only if $Y = aX+b$ exactly.
4. $X \perp\!\!\!\perp Y \Rightarrow \rho_{X,Y} = 0$.