Chapter 2

Basic Set Theory, Experiments and Events

Lecture 2 | Devore 9e: 2.1

Today there is still *no probability*. We only build the language: a random experiment produces outcomes, outcomes form a set, and events are subsets of that set. Every probability statement later in the course is a statement about sets, so it pays to be fluent here first.

2.1 Sets

Definition 2.1 (Set)**.** A **set** is a collection of *distinct* objects. The objects are its **elements**; we write $x \in A$ if x is an element of A, and $x \notin A$ otherwise.

Example 2.2. $\{\text{“apple”}, \text{“orange”}, \text{“car”}\}$, $\{0, 1, \pi, \frac{1}{2}\}$, $\mathbb{N} = \{0, 1, 2, \dots\}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$. A set can also be described by a property rather than listed:

$$\big\{(x, y) : 0 \le x \le 1,\ 0 \le y \le 1,\ x, y \in \mathbb{R}\big\}$$

is the unit square.

Definition 2.3 (Empty set)**.** The **empty set** $\emptyset$ is the set with nothing in it.

Remark 2.4. $\{\emptyset\} \neq \emptyset$. The left side is a set with one element (that element happening to be the empty set); the right side has no elements. A bag containing an empty bag is not an empty bag.

2.1.1 Relations and operations

Throughout, A and B are sets, and all of them sit inside a fixed universal set $\mathcal{U}$.

Definition 2.5 (Subset)**.** A is a **subset** of B, written $A \subset B$, if $\forall x \in A$ we have $x \in B$.

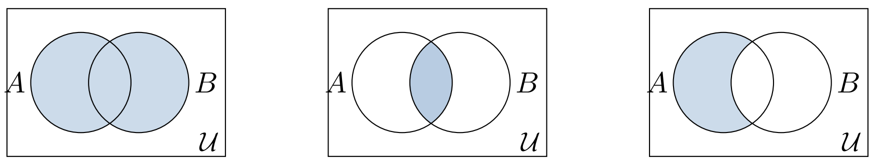
Remark 2.6. Some books write $A \subseteq B$ for this and reserve $A \subset B$ for a *proper* subset. In this course $A \subset B$ always allows $A = B$.

Definition 2.7 (Equality)**.** If $A \subset B$ and $B \subset A$, then $A = B$.

This is the standard way to prove two sets are equal: show each contains the other.

Definition 2.8 (Complement, union, intersection, difference)**.**

$$\begin{aligned} A^c = \overline{A} &:= \{x : x \notin A\} && \text{(complement)} \\ A \cup B &:= \{x : x \in A \textit{ or } x \in B\} && \text{(union)} \\ A \cap B &:= \{x : x \in A \textit{ and } x \in B\} && \text{(intersection)} \\ A - B = A \setminus B &:= \{x : x \in A \text{ and } x \notin B\} = A \cap B^c && \text{(difference)} \end{aligned}$$



$A \cup B$ $\qquad$ $A \cap B$ $\qquad$ $A - B$

Theorem 2.9 (De Morgan's rules)**.**

$$(A \cup B)^c = A^c \cap B^c, \qquad (A \cap B)^c = A^c \cup B^c.$$

In words: "not (A or B)" means "not A and not B". These are the identities that let us convert a hard union into an easy intersection, and we will use them constantly ("at least one" problems are almost always done by complementing to "none").

2.1.2 Cardinality

Definition 2.10 (Cardinality)**.** $|A|$ is the number of elements of A.

The three-way split below is exactly what forces the three definitions of probability in Lectures 3–4.

$\|A\| < \infty$	**finite**	$\{1, 2, 3, 4, 5, 6\}$
$\|A\| = \infty$	**countably** infinite	$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$
	uncountably infinite	$(0, 1)$, $\mathbb{R}$, the unit square

Remark 2.11. Countable means "can be listed as a sequence $a_1, a_2, a_3, \ldots$". Surprisingly $|\mathbb{Q}| = |\mathbb{Z}|$ — the rationals can be listed — while $(0, 1)$ cannot. A number like $\pi/10 \in (0, 1)$ is not on anybody's list.

2.2 Collections of sets

Definition 2.12 (Collection of sets)**.** Let Γ be an **index set**. A **collection** of sets indexed by Γ is written $\{A_\alpha\}_{\alpha\in\Gamma}$.

Example 2.13.

- $\Gamma = \mathbb{N}$: $\quad \{A_i\}_{i\in\mathbb{N}} = \{A_1, A_2, A_3, \dots\}$.
- $\Gamma = (0,1)$ and $A_\alpha = (-\alpha, \alpha)$: an uncountable collection of open intervals.

Definition 2.14 (Countable union and intersection)**.** For $A_1, A_2, A_3, \dots,$

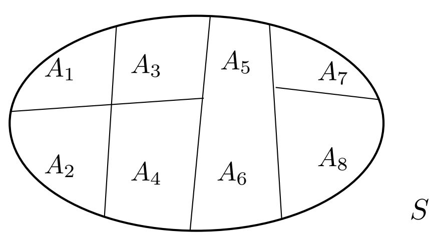
$$\bigcup_{i\in\mathbb{N}} A_i = \{x : x \in A_i \text{ for } \textit{some } i\}, \qquad \bigcap_{i\in\mathbb{N}} A_i = \{x : x \in A_i \text{ for } \textit{every } i\}.$$

Definition 2.15 (Pairwise disjoint)**.** $\{A_i\}_{i\in\mathbb{N}}$ is **pairwise disjoint** if $A_i \cap A_j = \emptyset$ for all $i \neq j$.

Definition 2.16 (Partition)**.** $\{A_i\}_{i\in\mathbb{N}}$ is a **partition** of a set S if

1. the A_i are pairwise disjoint, and
2. $\bigcup_{i\in\mathbb{N}} A_i = S$.

A partition cuts S into non-overlapping pieces that use up all of S. This is the single most useful picture in the course — the law of total probability (Lecture 6) is nothing but "compute on each piece of a partition, then add up".



2.3 Experiments, outcomes, events

Definition 2.17 (Experiment)**.** An **experiment** is a repeatable task with well-defined outcomes. Individual outcomes are denoted $\omega_1, \omega_2, \dots,$ and the set of all of them is the **sample space** S.

Definition 2.18 (Event)**.** Any subset $E \subset S$ is an **event**. We say E **happens** if the outcome of the experiment lies in E. An event with exactly one outcome in it is a **simple event**.

Example 2.19 (Finite sample space)**.** Roll a six-sided die once: $S = \{1, 2, 3, 4, 5, 6\}$.

- "You got an even number": $E = \{2, 4, 6\}$.
- "You got the number 7": $E = \emptyset$.

Roll a die, then toss a coin:

$$S = \{(1, H), (1, T), (2, H), (2, T), \dots, (6, H), (6, T)\}, \qquad |S| = 12.$$

Example 2.20 (Countably infinite sample space)**.** Record the number of whole minutes until the first call arrives at a call center:

$$S = \{0, 1, 2, 3, \dots\}.$$

Example 2.21 (Uncountable sample space)**.** Pick a point at random from the unit square:

$$S = [0, 1] \times [0, 1] = \{(x, y) : 0 \le x \le 1,\ 0 \le y \le 1\}.$$

The event "the point is within distance 1 of the origin" is $E = \{(x, y) \in S : x^2 + y^2 \le 1\}$, a quarter disc.

Example 2.22 (Uber Eats)**.** You order fries and a pizza. Let x be the delivery time of the fries in minutes and y that of the pizza, so an outcome is a pair and

$$S = \{(x, y) : x \ge 0,\ y \ge 0\}.$$

Now translate. Let

$A = \{(x, y) \in S : x \le 10\}$ (fries arrive within 10 min), $\qquad B = \{(x, y) \in S : y \le 15\}$ (pizza within 15 min).

Then

$A \cap B$	the whole meal is ready within 15 minutes
$A \cup B$	at least one of the two arrives on time
$A - B$	the fries arrive but you wait past 15 minutes for the pizza
$(A \cup B)^c$	neither arrives on time

Read the third row carefully: $A - B$ is the event that you have fries and nothing else for at least five minutes. Getting from an English sentence to the right set is the skill this lecture is for.

2.4 The dictionary

This table is the whole point of the lecture. Read it left to right when you are formalizing a word problem, and right to left when you are interpreting a formula.

Set notation	**Colloquial description**
S	sure event (something happens)
$\emptyset$	impossible event (nothing happens)
A	event A happens
A^c	A does not happen
$A \cup B$	A *or* B (or both)
$A \cap B$	A *and* B
$A \setminus B$	A happens but B does not
$A \cap B = \emptyset$	A and B are mutually exclusive
$\bigcup_{i \in \mathbb{N}} A_i$	at least one of the events happens
$\bigcap_{i \in \mathbb{N}} A_i$	all of the events happen

Remark 2.23. Note what is *not* on this page: any number between 0 and 1. We have a language for events but no way yet to say how likely one is. That starts next lecture.