# Chapter 3

# Classical Probability, Counting and Combinatorics

Lecture 3 | Devore 9e: 2.2–2.3

Last time: set theory, experiments, events — and no probability anywhere. Today we assign numbers to events for the first time, in the oldest and simplest way.

## 3.1 Classical (Laplace) probability

Roll a fair die. $S = \{1, 2, 3, 4, 5, 6\}$. What is "the probability" here? We assign each outcome a number, and symmetry says they should all get the same one:

$$\mathbb{P}(1) = \mathbb{P}(2) = \cdots = \mathbb{P}(6) = \tfrac{1}{6}.$$

**Definition 3.1** (Classical probability)**.** Let $S = \{\omega_1, \ldots, \omega_n\}$ be *finite* with all outcomes equally likely, $\mathbb{P}(\omega_1) = \cdots = \mathbb{P}(\omega_n) = \frac{1}{n}$. For an event $A \subset S$ containing $m$ outcomes,

$$\boxed{\mathbb{P}(A) = \frac{m}{n} = \frac{|A|}{|S|}}$$

The triple $(S, \text{“events”}, \mathbb{P})$ — where "events" means all subsets of $S$ — is our first probability model. Two things to notice:

- **Computing $\mathbb{P}$ is exactly counting.** There is no probability theory left in the formula; there is only $m$ and $n$.
- **Counting can be very tricky.** That is the real content of this lecture, and of most exam problems in the first month.

## 3.2 Two fundamental principles of counting

**Theorem 3.2** (Addition principle). *If a task can be done in $m$ ways or in $n$ ways, and the two groups of ways do not overlap, the total is $m + n$.*

**Theorem 3.3** (Multiplication principle). *If a task is done in two steps, the first in $m$ ways and the second in $n$ ways, the total is $m \times n$. With $k$ steps, multiply all $k$ counts.*

**Example 3.4** (Dining halls — addition). Three dining halls on campus offer 10, 8 and 6 entrées. If you will eat at exactly one of them, you have

$$10 + 8 + 6 = 24$$

choices. "Or" means add — and note this is only correct because no entrée is counted twice.

**Example 3.5** (Passcode — multiplication). A 4-digit phone passcode is built by choosing each digit in turn, 10 ways each:

$$10 \times 10 \times 10 \times 10 = 10^4 = 10{,}000.$$

"And then" means multiply.

**Remark 3.6.** "Or" $\Rightarrow$ add; "and then" $\Rightarrow$ multiply. Most counting errors come from using the wrong one, or from steps that are not independent of each other (the count at step 2 must not depend on *which* choice was made at step 1 — only on how many there were).

## 3.3 Permutations and combinations

The organizing question: *you have $n$ distinct objects and select $k$ of them. How many ways?* The answer depends on two yes/no questions:

Does **order** matter?
Is the selection **with replacement**?

That is four cases. We do all four.

### 3.3.1 Case 1: order matters, without replacement

Build the selection in $k$ steps:

$$\begin{array}{ll} \text{step 1:} & n \text{ choices} \\ \text{step 2:} & n-1 \text{ choices} \\ \vdots & \\ \text{step } k\text{:} & n-k+1 = n-(k-1) \text{ choices} \end{array}$$

**Definition 3.7** (Permutation). The number of ways to arrange $k$ objects out of $n$ distinct objects, order mattering, without replacement, is

$$A_n^k := n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!},$$

read "*n permute k*".

Taking $k = n$ gives $A_n^n = n!$: the number of ways to order all $n$ objects.

### 3.3.2 Case 2: order does not matter, without replacement

Here is the trick that makes this case free. Every unordered selection of $k$ objects can be ordered in $A_k^k = k!$ ways, so

$$\underbrace{A_n^k}_{\text{order matters}} = \underbrace{\big(\#\text{ order does not matter}\big)}_{\text{what we want}} \times k!.$$

**Definition 3.8** (Combination)**.**

$$C_n^k = \binom{n}{k} := \frac{A_n^k}{k!} = \frac{n!}{k!\,(n-k)!},$$

read "*n choose k*". Equivalently $A_n^k = \binom{n}{k} \times k!$.

**Remark 3.9** (Two notations)**.** I write $C_n^k$ on the board and $\binom{n}{k}$ in typed material; they are the same number. Likewise $A_n^k$ is often written $P(n,k)$ or $nPk$ elsewhere.

**Example 3.10.** Take $n = 4$ objects $\{1,2,3,4\}$ and $k = 2$. Ordered: $A_4^2 = 12$ pairs $(1,2),(2,1),(1,3),(3,1),\ldots$ Unordered: $\binom{4}{2} = 6$ sets $\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}$. And indeed $12 = 6 \times 2!$.

**Example 3.11** (Candy — the relation made concrete)**.** There are 10 different candies.

- *Pick* 3 *to eat for dessert.* You will eat all three, so the order does not matter: {red, green, blue} is the same dessert as {green, red, blue}. That is $\binom{10}{3} = 120$ ways.
- *Pick* 3 *and hand them to Alice, Bob and Carol.* Now it matters who gets which. Each of the 120 selections can be handed out in $3! = 6$ ways, so

$$\binom{10}{3} \times 3! = 120 \times 6 = 720 = A_{10}^3.$$

The second bullet *is* the identity $A_n^k = \binom{n}{k}\,k!$: choosing an ordered list is the same as choosing the set and then ordering it.

### 3.3.3 Case 3: order matters, with replacement

Each of the $k$ steps now has all $n$ objects available again:

$$\underbrace{n \times n \times \cdots \times n}_{k} = n^k.$$

### 3.3.4 Case 4: order does not matter, with replacement

This one is genuinely tricky, and the standard trick is worth knowing.

Since order does not matter, all that is recorded is *how many times each object was selected*. Say object $i$ was chosen $x_i$ times. Then we need the number of integer solutions of

$$\sum_{i=1}^{n} x_i = k, \qquad x_i \geq 0.$$

**The balls-and-bars model.** Draw the $k$ selections as $k$ balls in a row. To split them into $n$ groups we need $n-1$ bars:

$$\underbrace{\circ\circ}_{x_1}\;\Big|\;\underbrace{\circ\circ}_{x_2}\;\Big|\;\underbrace{\circ}_{x_3}\;\Big|\;\underbrace{\circ\,\circ\,\circ}_{x_4}$$

There are $n+k-1$ positions in total (balls plus bars), and choosing which $n-1$ of them are bars determines everything:

$$\boxed{\binom{n+k-1}{n-1}}$$

### 3.3.5 The summary table

| | **without replacement** | **with replacement** |
|---|---|---|
| **order matters** | $A_n^k = \dfrac{n!}{(n-k)!}$ | $n^k$ |
| **order does not matter** | $\dbinom{n}{k} = \dfrac{n!}{k!(n-k)!}$ | $\dbinom{n+k-1}{n-1}$ |

**Remark 3.12.** Memorize the table, but more importantly memorize the two *questions*. On an exam the hard part is never the formula — it is deciding which box you are in.

## 3.4 Poker: answering Lecture 1's question

A five-card hand is dealt from a standard 52-card deck (13 ranks $\times$ 4 suits). Order does not matter and there is no replacement, so we are in the lower-left box of the table:

$$|S| = \binom{52}{5} = 2{,}598{,}960.$$

Every hand is equally likely, so classical probability applies and each computation below is just "count the favorable hands".

**Royal flush** — $10, J, Q, K, A$ all in one suit. There is exactly one such hand per suit, so 4 hands:

$$\mathbb{P} = \frac{4}{2{,}598{,}960} = 0.000154\%.$$

**Straight flush** — five consecutive ranks, all one suit. A straight is determined by its lowest card, which can be $A, 2, 3, \ldots, 10$ (with $A$ playing low in $A2345$ and high in $10JQKA$), so there are 10 possible runs per suit and $4 \times 10 = 40$ straight flushes. Excluding the 4 royal flushes, which are already ranked above:

$$\mathbb{P} = \frac{40 - 4}{2{,}598{,}960} = \frac{36}{2{,}598{,}960} = 0.001385\%.$$

**Four of a kind** — build it in two steps. Choose the rank that appears four times (13 ways); all four of its cards are then forced. Choose the remaining fifth card from the other 48 cards (48 ways):

$$\mathbb{P} = \frac{13 \times 48}{2{,}598{,}960} = \frac{624}{2{,}598{,}960} = 0.024010\%.$$

**Full house** — three of one rank plus two of another. Four steps, multiplied:

$$\underbrace{13}_{\text{rank of the triple}} \times \underbrace{\binom{4}{3}}_{\text{which 3 suits}} \times \underbrace{12}_{\text{rank of the pair}} \times \underbrace{\binom{4}{2}}_{\text{which 2 suits}} = 13 \times 4 \times 12 \times 6 = 3{,}744,$$

$$\mathbb{P} = \frac{3{,}744}{2{,}598{,}960} = 0.144058\%.$$

Note the 12, not 13: the pair must be a *different* rank from the triple.

**Flush** — all five cards one suit. Choose the suit (4) and then 5 of its 13 cards, $\binom{13}{5} = 1287$. That gives $4 \times 1287 = 5{,}148$, but this count includes the 40 straight flushes, which rank higher:

$$\mathbb{P} = \frac{5{,}148 - 40}{2{,}598{,}960} = \frac{5{,}108}{2{,}598{,}960} = 0.196540\%.$$

**Straight** — five consecutive ranks, suits arbitrary. Choose the run (10 ways), then a suit for each of the five cards independently ($4^5 = 1024$), giving 10,240; again remove the 40 straight flushes:

$$\mathbb{P} = \frac{10{,}240 - 40}{2{,}598{,}960} = \frac{10{,}200}{2{,}598{,}960} = 0.392465\%.$$

| **Hand** | **Count** | **Probability** |
|---|---:|---:|
| Royal flush | 4 | 0.000154% |
| Straight flush | 36 | 0.001385% |
| Four of a kind | 624 | 0.024010% |
| Full house | 3,744 | 0.144058% |
| Flush | 5,108 | 0.196540% |
| Straight | 10,200 | 0.392465% |

**Remark 3.13.** There is the answer to Question 1 of Lecture 1. The ranking is not a convention: each hand beats the one below it because it is strictly rarer. A flush outranks a straight because there are 5,108 flushes and 10,200 straights — a straight is almost exactly twice as likely.