Chapter 13

Continuous Random Variables: pdf, cdf, Percentiles

Lecture 13 | Devore 9e: 4.1–4.2

13.1 What goes wrong with a pmf

A random variable $X : S \to \mathbb{R}$ is **discrete** if its values form a finite or countable set, and **continuous** if they fill intervals — $[a, b]$, $[a, \infty)$, or a union such as $(a, b) \cup (c, \infty)$.

Example 13.1 (Height in the Himalayas)**.** Pick a point at random in a mountain region and let X be its elevation, so $X \in [0 \text{ ft}, 29\,029 \text{ ft}]$. What is $\mathbb{P}\{X = 10\,000 \text{ ft}\}$?

Exactly $10\,000.000\ldots$ feet, to infinite precision, is a contour line of zero area. The answer is 0. And the same argument applies to *every* value:

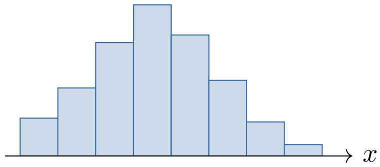
$$\mathbb{P}\{X = c\} = 0 \quad \text{for every } c.$$

So a pmf is useless here — it would be identically zero. But the variable is certainly not trivial: some elevation bands are far more likely than others. The information lives in *intervals*, not in points.

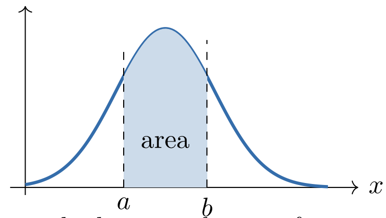
> **Definition 13.2** (Continuous random variable)**.** X is **continuous** if its possible values form intervals and $\mathbb{P}\{X = c\} = 0$ for every $c \in \mathbb{R}$.

From histogram to density

Chop the range into bands $[0, 100]$, $(100, 200]$, $(200, 300]$, ... and record the probability of each. That gives a histogram. Now refine the bands: each bar gets shorter, so instead plot *probability per unit length*, i.e. bar height divided by bar width. As the width goes to 0 the tops of the bars trace out a curve. That curve is the density.



coarse bands



the limiting density f

13.2 The probability density function

Definition 13.3 (pdf)**.** The **probability density function** of a continuous random variable X is a function f such that for all $a < b$,

$$\mathbb{P}\{a \leq X \leq b\} = \int_a^b f(x)\,\mathrm{d}x.$$

Theorem 13.4 (Properties)**.**

$$f(x) \geq 0 \quad \textit{for all } x, \qquad \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1.$$

Conversely, any f with these two properties is the pdf of some continuous random variable.

Two consequences worth stating separately.

Proposition 13.5. $\mathbb{P}\{X = c\} = 0$ *for every c, and therefore*

$$\mathbb{P}\{a \leq X \leq b\} = \mathbb{P}\{a < X \leq b\} = \mathbb{P}\{a \leq X < b\} = \mathbb{P}\{a < X < b\}.$$

Proof. $\mathbb{P}\{X = c\} = \lim_{h\to 0} \int_c^{c+h} f(x)\,\mathrm{d}x = 0.$ □

Remark 13.6 (The crucial warning)**.**

$$\boxed{f(x) \neq \mathbb{P}\{X = x\}}$$

A pdf is *not* a probability. It is a probability *per unit length*, so it can exceed 1 — $f(x) = 5$ on $[0, 0.2]$ is a perfectly good density. What f measures is relative likelihood: for small δ,

$$\mathbb{P}\{c - \delta \leq X \leq c + \delta\} = \int_{c-\delta}^{c+\delta} f(x)\,\mathrm{d}x \approx 2\delta f(c),$$

so a large $f(c)$ means X is likely to land *near* c, not *at* c.

Discrete	**Continuous**
values countable	values fill intervals
pmf $p(x) = \mathbb{P}\{X = x\}$	pdf $f(x) \neq \mathbb{P}\{X = x\}$
$\sum_i \mathbb{P}\{X = x_i\} = 1$	$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1$
$\mathbb{P}(A) = \sum_{x_i \in A} p(x_i)$	$\mathbb{P}(A) = \int_A f(x)\,\mathrm{d}x$

13.3 The cdf, again

The cdf is defined exactly as before, and it is what unifies the two cases.

Definition 13.7 (cdf of a continuous random variable)**.**

$$F_X(x) = \mathbb{P}\{X \leq x\} = \int_{-\infty}^{x} f(t)\,\mathrm{d}t.$$

Theorem 13.8. *Where f is continuous,*

$$f(x) = F_X'(x),$$

by the fundamental theorem of calculus. Moreover

$$\mathbb{P}\{a < X < b\} = F_X(b) - F_X(a), \qquad \mathbb{P}\{X > a\} = 1 - F_X(a).$$

Remark 13.9. Compare with the discrete case: there, F was a step function and the pmf was the size of the jumps. Here F is continuous and the pdf is its slope. Same object, two regimes — which is why Lecture 8 insisted that the cdf, not the pmf, is the fundamental description.

13.4 Percentiles

Definition 13.10 (Percentile)**.** For $0 < p < 1$, the $(100p)$-th percentile of a continuous random variable X is the value x_p with

$$\mathbb{P}\{X \leq x_p\} = p, \qquad \text{i.e.} \qquad F_X(x_p) = p.$$

So $x_p = F_X^{-1}(p)$: percentiles are what you get by reading the cdf backwards.

median	50th percentile
first quartile	25th percentile
third quartile	75th percentile

Example 13.11 (Percentiles of a uniform)**.** For $X \sim \text{Uniform}[A, B]$ we will see below that $F(x) = \frac{x-A}{B-A}$ on $[A, B]$. To find the 30th percentile, set $F(x_{0.3}) = 0.3$ and solve:

$$\frac{x_{0.3} - A}{B - A} = 0.3 \qquad \Longrightarrow \qquad x_{0.3} = A + 0.3\,(B - A).$$

So it sits 30% of the way along the interval — exactly what "uniform" should mean. The median is $x_{0.5} = A + \frac{1}{2}(B - A) = \frac{A+B}{2}$.

13.5 Two examples

Example 13.12 (Uniform distribution on $[A, B]$)**.** "Equally likely anywhere in $[A, B]$" means constant density:

$$f(x) = \begin{cases} \dfrac{1}{B-A}, & A \le x \le B, \\ 0, & \text{elsewhere}, \end{cases} \qquad F(x) = \begin{cases} 0, & x < A, \\ \dfrac{x-A}{B-A}, & A \le x \le B, \\ 1, & x > B. \end{cases}$$

The constant $\frac{1}{B-A}$ is forced by $\int f = 1$. Note it exceeds 1 whenever $B - A < 1$.

Example 13.13 (Throwing a dart — density is not uniform even when the dart is)**.** A dart lands uniformly on a disc of radius 1. Let X be the distance from the center. *The dart is uniform on the disc, but X is not uniform on* $[0, 1]$*.*

Go through the cdf, which is where geometric probability from Lecture 4 pays off:

$$F(x) = \mathbb{P}\{X \le x\} = \frac{\text{area of disc of radius } x}{\text{area of disc of radius } 1} = \frac{\pi x^2}{\pi \cdot 1^2} = x^2, \qquad 0 \le x \le 1.$$

Differentiate:

$$f(x) = F'(x) = 2x, \qquad 0 \le x \le 1.$$

Check: $\int_0^1 2x \, \mathrm{d}x = 1$. ✓ The density increases linearly, because there is more area far from the center than near it.

Remark 13.14. This example is the template for the change-of-variable method in Lecture 15: to find the distribution of a *function* of a random variable, compute its cdf first and differentiate at the end. Do not try to transform densities directly.

13.6 Expectation in the continuous case

Every sum becomes an integral and every pmf becomes a pdf:

$$\underbrace{\sum_i x_i \, p(x_i)}_{\text{discrete}} \qquad \rightsquigarrow \qquad \boxed{\mathbb{E}[X] = \int_{-\infty}^{\infty} x \, f(x) \, \mathrm{d}x}$$

$$\mathbb{E}[h(X)] = \int_{-\infty}^{\infty} h(x) f(x) \, \mathrm{d}x, \qquad \operatorname{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \int_{-\infty}^{\infty} x^2 f(x) \, \mathrm{d}x - \mu^2.$$

Remark 13.15 (Physical reading)**.** Since $\int f = 1$,

$$\mathbb{E}[X] = \frac{\int_{-\infty}^{\infty} x f(x) \, \mathrm{d}x}{\int_{-\infty}^{\infty} f(x) \, \mathrm{d}x},$$

which is precisely the **center of mass** of a rod with density f: the balance point of the pdf. This is a genuinely useful picture — for a symmetric density the mean is the axis of symmetry, no integration required.

Example 13.16. For the dart, $f(x) = 2x$ on $[0, 1]$:

$$\mathbb{E}[X] = \int_0^1 x \cdot 2x \, \mathrm{d}x = \frac{2}{3}, \qquad \mathbb{E}[X^2] = \int_0^1 x^2 \cdot 2x \, \mathrm{d}x = \frac{1}{2}, \qquad \operatorname{Var}(X) = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}.$$

The average distance from the center is $2/3$, not $1/2$.

13.7 A worked example start to finish

Example 13.17. A continuous random variable has cdf $F(x) = c\,x^2$ for $0 \le x \le 2$ (and the obvious 0 and 1 outside). Find c, the pdf, $\mathbb{E}[X]$ and $\mathrm{Var}(X)$.

Step 1: find c. A cdf must reach 1 at the top of the range, so $F(2) = 1$:

$$c \cdot 2^2 = 4c = 1 \qquad \Longrightarrow \qquad c = \tfrac{1}{4}, \qquad F(x) = \frac{x^2}{4}, \quad 0 \le x \le 2.$$

Step 2: differentiate to get the pdf.

$$f(x) = F'(x) = \frac{2x}{4} = \frac{x}{2}, \qquad 0 \le x \le 2,$$

and $f(x) = 0$ elsewhere. Check: $\int_0^2 \frac{x}{2}\,\mathrm{d}x = \frac{x^2}{4}\Big|_0^2 = 1$. ✓

Step 3: the mean.

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x)\,\mathrm{d}x = \underbrace{\int_{-\infty}^{0} x \cdot 0\,\mathrm{d}x}_{=0} + \int_0^2 x \cdot \frac{x}{2}\,\mathrm{d}x + \underbrace{\int_{2}^{\infty} x \cdot 0\,\mathrm{d}x}_{=0} = \frac{1}{2} \cdot \frac{x^3}{3}\Big|_0^2 = \frac{8}{6} = \frac{4}{3}.$$

Step 4: the second moment.

$$\mathbb{E}[X^2] = \int_0^2 x^2 \cdot \frac{x}{2}\,\mathrm{d}x = \frac{1}{2} \cdot \frac{x^4}{4}\Big|_0^2 = \frac{16}{8} = 2.$$

Step 5: the variance.

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \big(\mathbb{E}[X]\big)^2 = 2 - \left(\frac{4}{3}\right)^2 = 2 - \frac{16}{9} = \frac{2}{9}.$$

Remark 13.18. Step 3 is written out in full to make one point: the integral $\int_{-\infty}^{\infty}$ always splits at the edges of the support, and outside the support $f = 0$ so those pieces vanish. After you have done it once, just integrate over the support — but be sure you know *what* the support is. Note also the sanity check in step 2; a pdf that does not integrate to 1 means an error upstream, and catching it there costs nothing.